DEVELOPING MATHEMATICAL THINKING: GEOMETRY WORKBOOK

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DMT Workbook: Geometry K-8

This DMT Workbook: Geometry K-8 helps teachers and administrators develop a deep understanding of the structural components of geometry and related topics taught in Kindergarten through pre-algebra and in understanding the progressions of how students build a procedural and conceptual understanding of the mathematics over time. This workbook should provide you with a framework for how to promote these topics.

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INTRODUCTION

The focus of this geometry workshop is to enhance teachers' knowledge of mathematics, their understanding of how students' best learn mathematics and their ability to teach mathematics in a more effective manner. In addition, teachers will develop and use strategies to assist all students—high- and low-achieving—to understand mathematics and perform better on multiple achievement measures. More specifically, we focus on the importance of taking students' ideas seriously, encouraging multiple strategies, promoting conceptual understanding through discourse, focusing on misconceptions and understanding the key structural ideas of the relevant mathematics.

DEVELOPING MATHEMATICAL THINKING

Overview

Current research on learning has demonstrated that students should learn with understanding (Cohen, McLaughlin, & Talbert, 1993; Hiebert, 1986). Students need to understand topics in greater detail, become better problem solvers, and learn mathematics as an interconnected web of knowledge as opposed to isolated topics divorced from real-world or complex events (Romberg, 1992; Hiebert & Carpenter, 1992). More specifically, national organizations such as NCTM (1989, 1991, 1994, 2000), MSEB (1989, 1990, 1991), and NCSM (2010) recommend that students should learn to communicate and reason with mathematics, make connections within and outside of mathematics, and become problem solvers.

Teaching for Understanding (TfU)

Constructivist learning theories suggest students need to learn with understanding, a complex, dynamic process in which students connect pieces of knowledge to other related pieces of knowledge (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997). Carpenter and Lehrer (1999) describe understanding as knowing *how* to do something and *why*. Knowing how enables us to flexibly use procedures and understand the relationship of these procedures within the structure of the discipline. Knowing why enables us to use concepts flexibly, extend our knowledge to new situations, and connect it to the world outside of school (Ball, 1989; Hiebert et al., 1997; McDiarmid, Ball, & Anderson, 1989; NCTM, 2000; Newmann & Associates, 1996; Perkins, 1993).

Two useful and complementary perspectives of understanding are discussed in the literature: structural and functional. A structural perspective of understanding is defined as knowledge that is structured through web-like or hierarchal connections, or some combination of the two. More specifically, each topic or concept "is understood if its mental representation is part of a network of representations" (Hiebert & Carpenter, 1992, p. 67). The stronger and the greater number of connections there are in this complex structure, the higher degree of understanding there becomes (Hiebert & Carpenter, 1992).

A functional perspective of understanding, described in pragmatic terms, is how students interact and share knowledge with others (Hiebert et al., 1996). It is a social constructivist view of learning that maintains students need to have the chance to actively integrate incoming information with existing knowledge through social interactions (Wood, 1993). This suggests that by being in situations in which students are communicating with others, they are more able to construct a lasting and coherent appreciation of the concepts and skills they are learning. The ways in which students organize the information and the degree of their understanding is influenced by these situations as well.

Our concept of teaching for understanding is grounded in these two perspectives. From a structural position on understanding, our aim is toward creating conditions within students' classroom experiences that assist them in organizing new information in ways that allow them to order and build a well-connected network of understandings. From a functional perspective, we focus on providing the types of tasks and activities that place students in situations where, through articulation, they are able to reflect on how they solve problems and construct relationships.

In other words, in order for students to develop understanding, teachers must attend to both the structural and functional aspects of teaching the content within their classroom. As Carpenter and Lehrer argue, "For learning with understanding to occur, instruction needs to provide students the opportunity to develop productive relationships, extend and apply their knowledge, reflect about their experiences, articulate what they know, and make knowledge their own" (year, p. 32).

A Model for TFU: The DMT's "Instructional Theory"

Our model for teaching for understanding stems from the notions of "guided reinvention" and "mathematizing." (Freudenthal; 1973, 1991; Treffers, 1987). As Gravemeijer and van Galen (2003) describe, guided reinvention is a process of first allowing students to develop informal strategies for solving problems, and then, by critically examining those strategies, encouraging students to develop more sophisticated, formal, conventional and abstract strategies and algorithms. By comparing invented solution strategies, students learn which manipulations make sense for given contexts and are encouraged to develop more general procedures. Eventually, the contexts fade into the background and the "manipulations themselves … acquire meaning of their own" (Gravemeijer & van Galen, 2003, p. 116).

Through solving novel problems and examining multiple approaches, students make connections between existing knowledge (informal ideas) and new knowledge (more formal mathematical ideas), strengthening their structural perspective or connections. By critically examining their own and others' strategies, students are encouraged to build functional understanding which exemplifies the importance of social interactions in classrooms. "By thinking and talking about similarities and differences between arithmetic procedures, students can construct relationships between them. … the instructional goal is not necessarily to inform one procedure by the other but, rather, to help students build a coherent mental network in which all pieces are joined to others with multiple links" (Hiebert & Carpenter, 1992, p. 68).

Closely related to guided reinvention, our model also incorporates Treffer's (1987) notions of horizontal and vertical mathematization. Horizontal mathematization occurs when students represent a contextualized problem mathematically in order to find a solution strategy. Vertical mathematization involves reorganizing mathematical understanding by recognizing connections and shortcuts. This takes the mathematical matter to a higher level of abstraction and efficiency, and occurs when students make their representations and strategies objects of mathematical examination—a process of reification. Mathematizing covers such activities as generalizaing, justifying, formalizing, and curtailing – including, but not limited to, developing an abstract algorithm (Gravemeijer & van Galen, 2003). By focusing on both types of mathematizing in their classrooms, teachers must maintain a focus on the inherent structure of the mathematical ideas that are emerging. In addition, they must address students' misconceptions as they arise so these misconceptions do not hinder the mathematizing progression. One of the results of mathematizing is that teachers connect students' informal strategies, many of which may be developed outside of school, with more formal mathematical ideas. "One would predict that if children possessed internal networks constructed both in and out of school, and if they recognized the connections between them, their understanding and performance in both settings would improve" (Hiebert & Carpenter, 1992, p. 79). Each session addresses and highlights horizontal and vertical mathematizing.

Such a process starts with carefully chosen contextual problems. To solve these problems, students must model the situation to some degree. Rather than beginning with the standard algorithms and attempts to concretize them, teaching begins with students' common sense solutions to contextual problems that are real for them. By reflecting on the solution procedures they have used, students develop more sophisticated models and procedures that they can also use in other situations (Gravemeijer & van Galen, 2003, p. 114). In other words, teaching starts with considering students' ideas about solving real world problems.

Mathematical knowledge originates from students' attempts to model contextual situations. Models then become the basis for solving related problems as well as a means for support for more formal mathematical reasoning (Gravemeijer & van Galen, 2003). As Cobb (2000) described, this use of modeling "…implies a shift in classroom mathematical practices such that ways of symbolizing developed to initially express informal mathematical activity take on a life of their own and are used subsequently to support more formal mathematical activity in a range of situations" (p. 319). In this way, modeling is a fundamental process in learning mathematics. However, this view of models and modeling contrasts with current practices in mathematics instruction in which models are used to "concretize expert knowledge" (Gravemeijer & van Galen, 2003, p. 118) and contextual problems are presented only after students have mastered traditional ways of solving problems. In this way, guided reinvention and mathematizing via the use of models, *turn the focus toward students' ways of using models rather than on teacher or curriculum created ways of using models.*

Through enacting aspects of "guided reinvention" and "mathematizing" teachers develop a classroom practice that is based on the tenets of teaching for understanding. We believe their practice hinges on *five key elements of classroom practice*: the centrality of students' ideas, encouraging multiple solution strategies and models, pressing students conceptually, addressing misconceptions and maintaining a focus on the structure of the mathematics. By focusing their teaching practice on these five key elements, teachers shift their attention toward students' informal strategies for solving problems and the mathematical connections between multiple solution strategies as well as formal solution strategies. By encouraging students to use informal knowledge and experiences, student misconceptions are bound to arise. By acknowledging and addressing them, teachers encourage students to make sense of and correct their flawed ways of thinking, rather than glossing over them or ignoring them completely. The five key elements of Developing Mathematical Thinking (DMT) grow out of the concepts that mathematics exists of underlying and inherently related constructs and that students learn mathematics through creating web-like or hierarchical structures for these constructs (see figure 1).

Figure 1. DMT Framework

DMT Element 1: Taking Students' Ideas Serious

Mathematics pedagogy should be built on the experiences of the students. When students solve unfamiliar tasks that they don't readily know the solution path to, they begin solving it using ineffective or informal methods. Their ability to communicate their ideas or notate it mathematically is typically quite primitive, which then limits their ability to organize the material in a generalizeable way. However, by allowing students to grapple with these ideas, we are now able to assess whether they have more or less formal ways of solving the problem. In other words, we have just placed students in the process of acquiring the knowledge themselves. Now to capitalize on this mental state, teachers must push students to use more formal models to describe, organize, and then extend their mathematical thinking.

DMT Element 2: Pressing Students Conceptually

One issue with more traditional instruction is that when teachers attempt to concretize and explain traditional algorithms through manipulatives or visuals that "represent" the procedures of the algorithm, students still fail to show mastery and struggle to apply algorithms in applied situations (Schoenfeld, 1987; Gravemeijer & van Galen, 2003). In contrast, when instruction moves toward pressing students to understand the procedural and conceptual aspects of the mathematics, students begin creating a more intuitive and detailed framework that in return, enables them to support their ideas when rules are forgotten (de Lange, Burrill, Romberg & van Reeuwijk, 1993). This framework is constantly built on previous knowledge in more and more structured ways, which allows students to continually extend and advance their current thinking, bounding and defining how and when to use the mathematics. Thus, once students' ideas have been generated by solving novel tasks, teachers need to ask questions such as, "How did you solve the problem?" "Will that solution path work for other types of problems or numbers?" In sum, understanding how and why a particular mathematical procedure, representation or model works, helps students retain the process and provides a strong foundation on which to transfer ideas to similar and unique situations.

DMT Element 3: Encouraging Multiple Strategies

One approach to deepening students' understanding and ensuring they have a rich mathematical foundation on which to move forward is to encourage students to work on problems from different vantages and/or use different mathematical models to solve the problem. The power of working with multiple models for students is that each model highlights a different aspect of the mathematics, thus building a richer knowledge base. In addition, when teachers require students to look "across problem contexts and solution methods for commonalities and differences" (Smith, 2003, 263), their students begin to generalize and move to more formal representations. Instruction can now encourage justification and proof by establishing an expectation that solution strategies be mathematically reasonable and justifiable (Ball & Bass, 2003; Silver, Leung, & Cai, 1995; Campbell, Rowan, & Suarez, 1998). This approach also becomes an opportunity to engage students in the mathematics by examining "the validity of alternative approaches" (Cai & Kenny, 2000).

It is our job as teachers to conceptualize possible learning trajectories through the use of multiple solution strategies. This enables us to make decisions on which pedagogical actions are best at a given time (near or far) for promoting mathematical understanding with a student or students. Bruner's three modes of representation – enactment, iconic and symbolic – are a helpful framework to help promote learning progressions.

DMT Element 4: Address Misconceptions

The work of developing constructive conceptions and mitigating misconceptions is an important aspect of teaching. Cognitively, it is a view of learning that holds that students actively integrate incoming information with preexisting knowledge that is already structured in some way. Students use this existing knowledge to make sense of incoming ideas. Understanding increases when children continually find or create relationships between new incoming information and existing schemata. "Without schema into which new information can be assimilated, experience is incomprehensible and therefore, little can be learned from it" (Romberg, 1993). From this position, it is critical that teachers create an instructional environment that addresses misconceptions and builds correct conceptions. The pedagogical actions must assist students in organizing new information so as to build well-connected schemata. When instruction ignores existing knowledge, understanding becomes more localized and restricted in use (Konold, 1993).

DMT Element 5: Focus on the Structure of the Mathematics

Finally, teaching must be framed around central or core structural ideas in the mathematics. The MTI course focuses on topics within number and algebra. These structural components are highlighted within each of the theoretical sections below for number, addition, subtraction, multiplication, division, decimals, fractions, ratio and proportion, and algebra. Some of these big ideas within each of these topics are decomposing, compensation and unitizing.

DMT Instructional Goals

Many conceptual models for learning and teaching refer to *hypothetical learning trajectories* (HLTs) or *learning progressions* (Baroody, et al, 2004; Clements & Sarama, 2004; Gravemeijer, 1999; Hiebert, et al, 1997; Simon, 1995). Generally, an HLT describes the mathematical path that teachers envision their students taking as they explore specific mathematical domains. HLTs include models of children's initial ideas, a sequence of instructional tasks, and descriptions of children's progressions of learning and thinking.

We attempt to prepare teachers to explore the terrain of mathematics in new ways. We encourage them to examine "connections among concepts, their representations and the various contexts in which they may be embedded" (Shifter, 1996, p. 3). As part of this process, it is crucial that teachers begin to question their own (typically traditional in nature) mathematical knowledge and the knowledge (based on real experiences) that their students bring to the classroom. We do this by not only encouraging teachers to enact the five key elements described above in their classroom practice, but we also help them learn to develop an understanding of possible *learning trajectories or learning progressions* (Gravemeijer & van Galen, 2003; Simon, 1995) for various mathematical topics.

According to Simon (2005), HLT's are made up of three components: the learning that defines the direction, the learning activities and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. Moreover, Baroody, et al (2004) emphasize that HLT's should focus on the "big ideas" and should incorporate both linear, or "ladder-like" trajectories, as well as multiple path or "branching tree" trajectories (p. 254). Finally, Hiebert, et al (1997) describe hypothetical learning trajectories as "…the teacher's vision of the mathematical path that the students might take, and its hypothetical nature comes from the fact that it is based on the teacher's guess about how learning might proceed along the path. The trajectory guides the teacher's task selection, but feedback from students and the teacher's assessments of the residues that are being formed lead to revisions in the trajectory. Tasks are selected purposefully, but the sequence can be revised" (p. 34).

As Gravemeijer (2004) points out, HLTs are tailored to a specific classroom and teacher at a given time. (For this reason, developing HLTs for general use may be a misuse of the HLT concept.) Gravemeijer (1999) develops another concept, *local instructional theory*, to refer to more general instructional sequences that may be useful on a larger scale. "The idea is that teachers use their insight in the local instruction theory to choose instructional activities and to design HLT for their own students. In my view, LITs can never free the teachers from having to design HLTs for their own classrooms. Nevertheless, I would argue that using a local instruction theory as a framework of reference could enhance the quality of the learning trajectories" (Gravemeijer, 2004, p. 107-108). This is ultimately the goal of the DMT and MTI professional development: to help teachers learn to utilize LITs in the creation of learning progressions that fit their particular students and classrooms.

SESSION 1: INTRODUCTION TO TEACHING FOR UNDERSTANDING

> One important purpose of mathematics education is to prepare students to incorporate mathematical reasoning and communication into their everyday lives. However, conventional pedagogy has often persuaded students to consider school mathematics as a subject divorced from their everyday experiences and from their attempts to make sense of their world (Tate, 1994).

> We construct our **knowledge** of our world from our perceptions and experiences, which are themselves mediated through our **previous knowledge.** Learning is the process by which human beings adapt to their **experiential** world (Simon, 1995)

The professional development goals are to encourage you to provide instruction that:

- Uses **reasoning** to make sense of problems;
- Moves from **informal** idea to more **formal** and abstract ones;
- Uses **articulation** of one's ideas and conjectures as a focal point to improved understanding.
- Understands the **generative** process of students' mathematical ideas and how to encourage fluid growth of these ideas.

In an international assessment of mathematics performance conducted in 40 countries across the world, the US ranked 28 in 2003 (PISA)

In 2009, 65 countries participated across the world, and the US ranked 31, which was below the OECD average (PISA)

Interest in math: math majors at four-year colleges has declined nearly 20% over the past eight years (Boaler, 2008)

Research indicates that teachers are the single most important factor in the success of their students (Loucks-Horsley, et al, 2003) .

A teacher of mathematics has a great opportunity. If he fills allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of the students by setting them to solve problems with *stimulating questions*, he may give them a taste for, and some means of, independent thinking (Polya)

Mental Math Activity

After you've solved them, record your strategy using paper & pencil. Try and avoid using any traditional procedures ("algorithm") to check your answer.

Task 1

Task 2

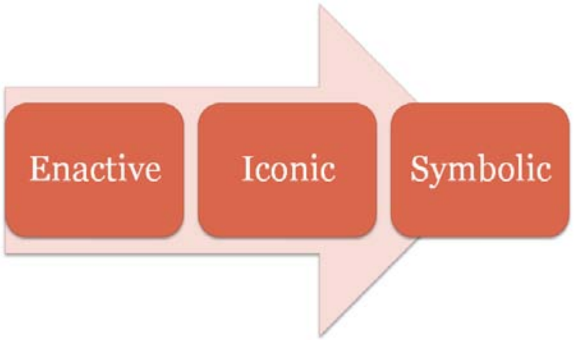
Chocolate Milk Task

One glass of milk has 1/3rd chocolate syrup. Another glass that is twice as large as the first glass contains ¼th chocolate syrup. If you put them together, how much syrup is in the new mixture?

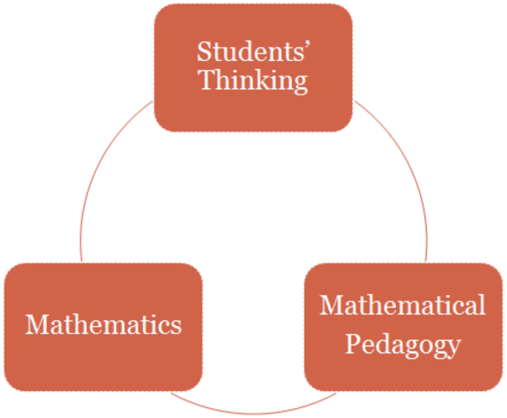
Extension:

Try to come up with an enactive, iconic, and symbolic representation or model.

Modes of Representation



Jerome Bruner (1966)



Teachers must know not only the formalism of mathematics but the informal and generative process of students' mathematical ideas and how to encourage fluid growth of these ideas.

By thinking and talking about similarities and differences between arithmetic procedures, students can construct relationships between them. … the instructional goal is not necessarily to inform one procedure by the other but, rather, to help students build a coherent mental network in which all pieces are joined to others with multiple links.

GEOMETRY FRAMEWORK

> "Geometry and spatial reasoning are inherently important because they involve 'grasping . . . that space in which the child lives, breathers and moves . . . that space that the child must learn to know, explore, conquer. In order to live, breathe and move better in it (Freudenthal, 1973, #)"

The Geometry Framework is built on the theoretical foundation of DMT with the main goal to improve mathematics instruction. Our central tenets match the philosophy of both the National Council of Teachers of Mathematics [NCTM] and the Mathematical Sciences Education Board [MSEB]. Their recommendations are that students learn to communicate and reason with mathematics, become better problem-solvers and make connections within mathematics and to other disciplines. However, it is not always clear how these learning goals translate into classroom practice. In this section, we propose the instructional model for teaching geometry that will serve as the foundation for the professional development work.

Geometry – the early years

Geometric shapes are all around us in the buildings we live and work in, the cars we ride in, even the food we eat. Children recognize these shapes early even if they do not know the technical terms for the different shapes they see. Geometry in the early elementary grades can build on children's informal knowledge by giving them appropriate vocabulary and opportunities to practice identifying geometric shapes in the classroom and the real-world. These learning activities help students make connections between what happens in and out of the classroom.

Space

Appropriate activities also help children begin to informally develop their concept of spatial sense through such observations as "close" and "far away." Giving students opportunities to practice developing their spatial sense will increase their awareness and application of geometric concepts in different areas of mathematics and other disciplines, including art, science, and social studies.

Geometry is an exciting part of mathematics for young children. As they are introduced to blocks, they begin their journey to understanding five key geometric dimensions: space, location, transformation, visualization, and justification or proof. One way to begin developing children's understanding of geometry is to provide different types of manipulatives for them to play with.

Blocks are very universal and enable students' to address all of these dimensions and become a large part of children's geometric development. There are many different types of blocks, linking and non-linking, foam and wooden, small and large. Each type of block affords children different experiences. Initially, children should play with large, non-linking, foam blocks. With these attributes children are free to experience the physical forces on blocks as they attempt to stack them, usually precariously. In addition, teachers should ask the children a set of questions regarding the blocks and, at the same time, introducing mathematical vocabulary. They might ask children to count the blocks as they stack them (0, 1, 2, . . .) and as they pull the construction apart (4, 3, 2, 1, 0).

Another important type of manipulative is puzzles. When children are introduced to objects where they must rotate and flip (reflect) before they fit, they begin to build spatial sense – especially visualization. We go deeper into this topic later.

As stated earlier, children should be asked mathematical questions and be introduced to mathematical vocabulary as they are playing with the blocks and puzzles. When they do so, they begin to construct mathematical ideas that are meaningful to them – thus enhancing long term memory and deepening understanding.

Shape

As was just described, children first begin to explore space in both two- and three-dimensions. In addition to space, shape is the other major topic of geometry. Initially, students are asked to identify shapes like, triangles, squares, rectangles, pyramids, and spheres. However, by only naming the shapes children will not build conceptual understanding. They also need to compare, sort, and describe shapes by focusing on their attributes. Children can create misconceptions unless their ideas are challenged using conjectures and examples and non-examples. For instance, a cube is not a good choice when discussing a square even though each face is a square because students may walk away thinking a square has thickness.

Here is an example to build understanding in geometry and an approach that can be used with other topics. Also note that this lesson builds understanding for the spectrum of students from disadvantaged to gifted students. First, ask children to find or draw all the different types of triangles they can. Next, have the students get up from their desks and do a gallery-walk – or walk in a continuous fashion around the room examining what others students have found or drawn. The students can now draw additional triangles that they might not have noted before and edit their previous drawings. This initial activity allows you to assess what students know about drawing triangles.

Next, ask the students to produce conjectures based on their drawings. (After looking for patterns in the data – in this case, the different triangles – the students can state or write down what they think are true statements. These statements are called conjectures and are meant to be proven true or false.) For instance, a student might say that "all triangles have a flat bottom" – indicating that the base of any triangle must be drawn along an x-axis or horizontal line. Other students should be asked to justify with examples or non-examples whether this conjecture is true. By writing, stating, and justifying conjectures, the students learn to use the correct mathematical language, understand that mathematics is a process of argumentation and problem solving, and begin to eliminate misconceptions while deepening their understanding of geometry.

Another activity to get students to develop geometric thinking is called Name that Shape. In this game, you or other students can give attributes of a shape to draw or build. Here is an example: This shape has four vertices or points connected with straight lines. What is it? Encourage students to think globally and determine whether there is only one solution or multiple solutions. Ask questions such as, what shape did you make? Explain how you created it. Are there any other solutions? What would happen in two-dimensions? What about three-dimensions? By asking these types of questions, students might draw different quadrilaterals. The idea is to encourage students to visually understanding shape and to be able to manipulate shapes in different ways to analyze the important elements. Allow students to use their own language while providing them with more formal or technical language.

Geometry – the later years

The teaching of geometry has changed over time. Historically, geometry has always been held as a critical method for examining and describing phenomenon and making logical arguments. Up until the 1960s, geometry always held a dominant percentage of the curriculum (which refers to all of their school experiences) in both elementary and secondary mathematics. Students spent up to 30% of the year working on geometry topics. Then, "advances in algebra strongly influenced the curricular reforms that began in the 1960s" (Jones & Mooney, 2003, 4). Geometry topics were reduced to less than 10% in most curriculum guides and textbooks. However, due to the current advances in technology (e.g., computer animation, GPS's medical imaging), geometry topics have had resurgence and many educators have pushed to increase the amount of time and topics with which students are confronted (Jones & Mooney, 2003).

Before we begin introducing more geometry into the curriculum, we need to understand what geometry is, how it has been presented to students and teachers, and how educators are going to be asked to present it in the future. In 1988, Battista and Clements described the current status of geometry: students performed poorly in geometry at the elementary levels. This was in part due to the reduction in geometric topics and due to the reliance and focus "on recognizing and naming geometric shapes and learning to write the proper symbolism for simple geometric concepts" (Battista & Clements, 1988, 11). At the time, mathematics educators were pressing to change elementary geometry to include "the study of objects, motions, and relationships in a spatial environment" (11). This change appeared to be radically different to most teachers and parents than focusing on memorization of shapes and their attributes. Educators were being asked is to build students' spatial reasoning.

Spatial Reasoning

Promoting spatial reasoning has never been a new proposition. Both mathematics educators and mathematicians over time have recognized the power and usefulness of having spatial reasoning (Casey, Andrews, Schindler, Kersh, Samper, & Copley, 2008; Jones & Mooney, 2003). As cited by Jones and Mooney in 2003, a renowned UK mathematician – Sir Michael Atiyah (2001, 50) described the importance this way:

> *Spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics – not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...*

Besides mathematicians, countries such as Singapore and the Netherlands (both of which perform well on the TIMSS and PISA) as well as other countries focus on the iconic representations or mathematical diagrams (such as the bar model and the number line) to visualize, understand, and solve problems (ben Haar, 2010; Casey et al, 2003; van den Heuvel-Panhuizen, 2008). So, if spatial reasoning is critical to mathematicians, employers, thinking, and solving problems, then what is it and how do we build it?

Building Spatial Reasoning

Spatial reasoning includes three elements: spatial visualization, spatial orientation, and spatial relations. The first element – *spatial visualization* – includes the ability to visually or "mentally manipulate, rotate, twist, or invert pictorially presented stimuli" (McGee, 1986, 28). The second element is *spatial orientation*, which involves the ability to remain unconfused when the object's orientation changes (Lee, 2005; McGee, 1986). And the third element – *spatial relations* – refers to the ability to recognize spatial patterns, understand spatial hierarchies, and imagining maps from verbal descriptions (Lee, 2005).

Having spatial sense helps when trying to make sense of complex geometric formulas in the upper grades. For instance, the formula for the area of a triangle, $A = \frac{1}{2}bh$, is far easier to make sense of and remember when a student can mentally 'copy' the triangle and manipulate the copy to join it with the original triangle to create a rectangle. In addition, spatial reasoning helps develop fluency with flexible operations in arithmetic and strengthens and supports students' ability in measurement. As with measurement, spatial reasoning also builds concepts of proportional reasoning, which aids the student in areas as diverse as geometry and data analysis.

In addition, spatial reasoning has a very high predictive value for mathematics achievement (Clements, 2008; Gustafsson & Undheim, 1996; Lubinski & Dawis, 1992). It is important to note that spatial ability is not related to quickness and, therefore, should not be timed (Carroll, 1993). And, although spatial reasoning items tend to favor males in upper grades, this is most likely related to opportunities than any innate skill (Caplan et al. 1985; Chadwick 1978; Fennema 1975; Harris 1978; Hyde et al. 1975; Jones & Burnett, 2006; Kali & Orion 1996; Linn and Petersen 1985; McGee 1979; Smith 1964).

Details of the Structural Components of Geometry

Before discussing specific standards or topics to teach at specific grade levels, it is important to have in mind the structural components of geometry. In other words, if you were asked, "What are three or four of the most important topics in geometry that span, if not all, most grade levels?" what might you say? We propose the structural components to be space and shape. "We must be able to understand the properties of objects and the relative positions of objects; we must be aware of how we see things and why we see them as we do; and we must learn to navigate through space and through constructions and shapes. This requires understanding the relationship between shapes and images (or visual representations)" (de Lange, 2001). Next, we describe these components in greater detail.

Space

Describing the world around us is one of the first experiences we have as humans. We do this by measuring, maneuvering, and understanding boundaries.

Measuring

- Measuring is a way of describing space that involves making comparisons between objects, locations, and units of measure.

Maneuvering

- Transformations
  - Movement paths, reflections, translations, and rotations
- Locations
  - Orientation, coordinates, directions, and comparative locations (e.g. nearer, above, inside, etc.)

Boundaries

- Defining or Describing Space (including visualization)
  - What are all of the possibilities in the space we are describing? What are the limitations?
- Dimensions
  - 1-D, 2-D, and 3-D

Shapes

Attributes and properties

- Attributes can be measured, properties are commonalities among a given class of shape or object

Composing/Decomposing

- Splitting a shape, using shapes to construct another shape, transferring from one dimension to another (e.g. squares becoming a cube)

Congruency and similarity

- Growing and Shrinking
    - How can an object or figure change in size but not in shape?
- Transformation
    - What does the position or orientation mean to the properties and attributes of a figure?

SHAPE TASKS

Triangle Recipe

1. Write down the steps (or recipe) for drawing a triangle.

2. Do your steps create a specific type of triangle or all types of triangles?

3. What are all the different types of triangles? What are their properties?

Triangle Name	Iconic Representation(s)	Attributes/Properties

Attribute Activity

Triangle Conjectures

Write at least 2 conjectures regarding triangles.

Determine whether the conjecture is true or false.

- Transfer your conjecture to an index card
- Put your initials on the bottom of your card
- Pass your conjectures to another person/group at a different table
- Read the conjecture and prove whether it is true or not

Classroom note:

- *Write down as many of the conjectures as you can. Give them to your students to determine whether they are true or false.*
- *Write down the different ways to prove or disprove the conjectures. Which ones are more informal and more formal?*

Quadrilateral Activity

What are all the different types of quadrilaterals? What are their properties?

Quadrilateral Name	Iconic Representation(s)	Attributes/Properties	Quadrilateral Name	Iconic Representation(s)	Attributes/Properties

Create a taxonomy of all the quadrilaterals

1. Name each of the quadrilateral
2. Place the most general type near the top and the most specific near the bottom
3. Use lines to show the relationship of each type

Extension: Write down as many conjectures as possible about quadrilaterals. (E.g., There are only three types of quadrilaterals that can have four congruent sides.)

Circle Activities

Task 1

1. Using only a piece of string, find the ratio of the distance around a circle to its largest chord. Do this for the three circles (Template 1 – Circles) or find three circles near you.

Circle	Distance around the circle	Distance of longest Chord	Ratio

2. Write a conjecture about the ratio of the distance around and any circle to its largest chord.

3. Test your conjecture (using measuring devices and calculations).
   a. What is the name of the distance around the circle?
   b. What is the name of the largest chord?
   c. What is the ratio of these two distances called?
   d. How can you use this information to determine a formula for the distance around the circle?

Task 2

Directions

1. Choose the largest circle and cut it out.

2. Fold the circle in half and half again. (Try to cut each of these sections in half again and then one more time.)

3. Cut the ‘pie’ shape pieces out and lay them next to each other.

Starting with the formula for the area of a parallelogram, create the formula for the area of circle.

Composing Shape

Tangram Activity

A Tangram is an ancient, unique, Chinese puzzle consisting of seven (geometric) pieces: 1 square, 5 triangles (2 large, 2 small, and 1 medium) and 1 parallelogram.

1. Focusing on area, what are the relationships among the geometric pieces?

2. Focusing on angle measurement, what are the relationships among the geometric pieces?

3. Cut out the pieces of the tangram and arrange them into different shapes or animals

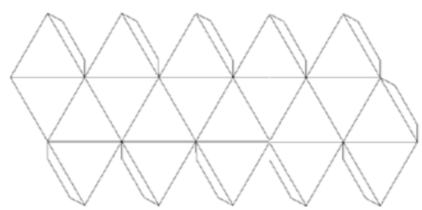
Composing Shape – Nets

A “Net” is a two-dimensional layout of a three-dimensional polyhedron.

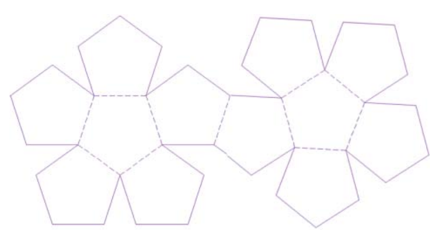
Building Nets

1. Use a circle compass, a ruler, (compass), and paper to create a net for the following polyhedrons:
    a. Tetrahedron
    b. Pyramid
        i. Square base
        ii. Pentagonal base
        iii. Hexagonal base
    c. Prism
        i. Square base or cube
        ii. Pentagonal base
        iii. Hexagonal base

2. Extension
    a. Icosahedron



    b. Dodecahedron



Building Formulas

1. In your group, record the number of faces, cubes, and edges for up to six different polyhedrons:

Polyhedron	Type of Face	No. of Faces	No. of Vertices	No. of Edges
Tetrahedron				
Hexahedron or Cube				
Octahedron				
Pentahedron or Square Pyramid				
Pentahedron or Triangular Prism				

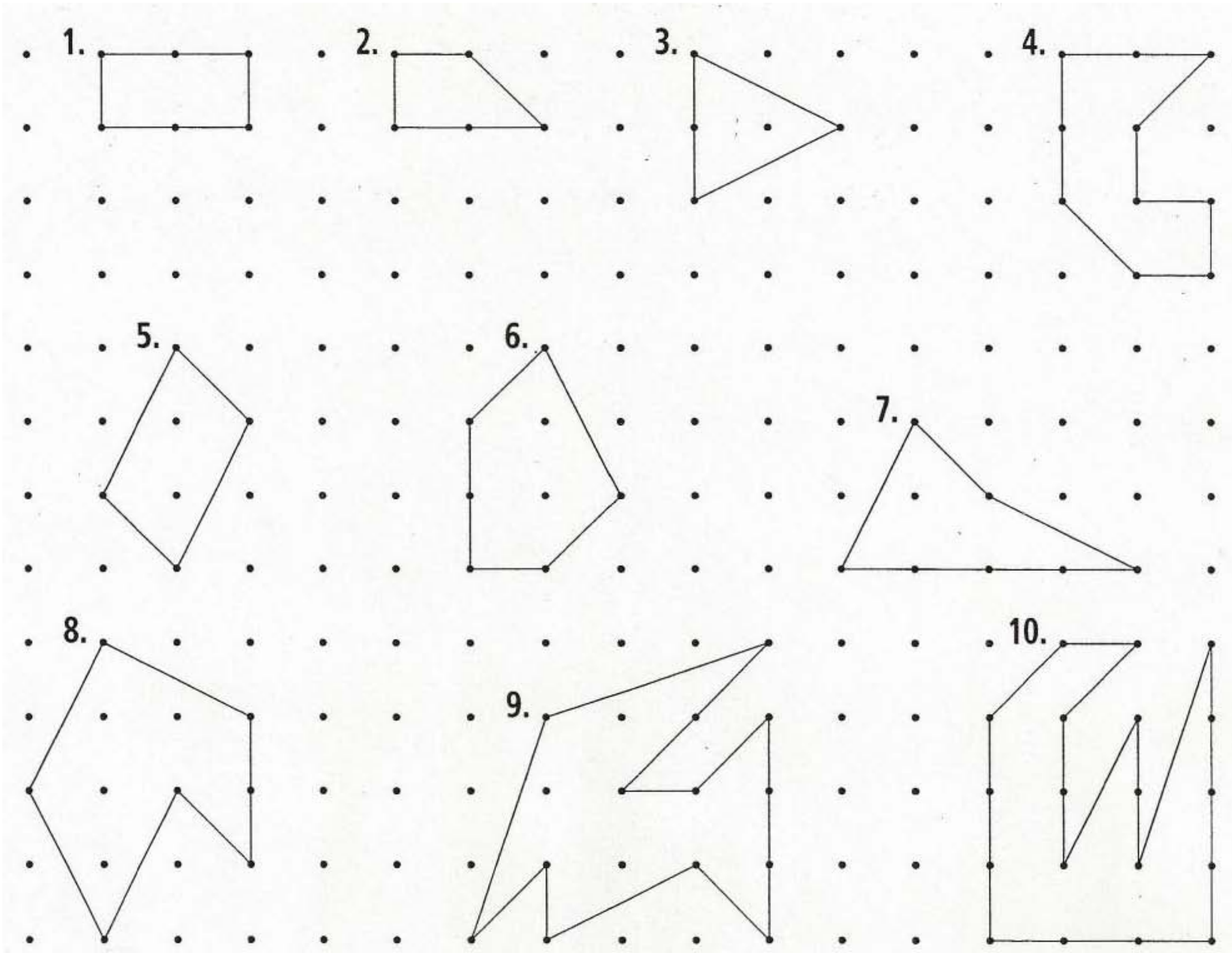
2. Make a conjecture based on the data in the table and then build a formula.

SPACE TASKS

> Poor performance of primary age pupils in geometry 'is due, in part, to the current elementary school geometry curriculum, which focuses on recognizing and naming geometric shapes and learning to write the proper symbolism for simple geometric concepts'. To remedy this situation they argue that geometry at the primary level should be 'the study of objects, motions, and relationships in a spatial environment' (Battista & Clements, 1988, 11).

Negative Space Task

Find the areas of the polygons labeled 1 – 10 below. Be prepared to describe your strategies.



From Dale Seymour

Lengths and squares

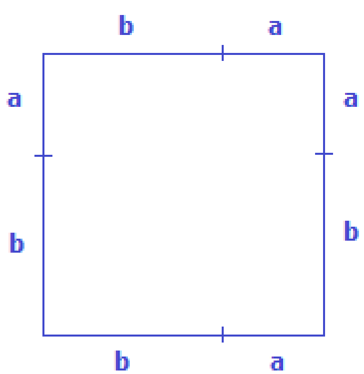
Use the dot paper to fill in the following table. Make a conjecture about your findings.

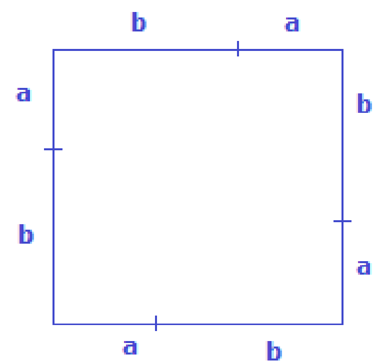
Length of Leg 1	Length of Leg 2	Area of square on Leg 1	Area of square on Leg 2	Area of square on the hypotenuse
1	1	1	1	2
1	2			
2	2			
1	3			
3	3			
3	4			
A	B			

Pythagorean Theorem ($a^2 + b^2 = c^2$)

1. Draw a right triangle (using the graph paper – Template 9) Label the short leg a, the long leg b, and the hypotenuse c.

2. Construct two squares (side by side) whose sides have length a + b.





3. Dissect the squares by drawing lines for the triangle you drew.

The Ramp

A carpenter wants to build a handicap ramp over a set of steps that is 12 feet long and 5 feet high. How long will the ramp be?

Extension:

Measuring Distances

Building a paper airplane

1. Build one of the two paper airplanes (see templates) and then complete the tasks on the next page.

2. Before flying your airplane, answer the following questions as best as you can (without conducting any measurements).

    a. How far can your airplane fly if it thrown from an altitude of 8 feet?

    b. What is the steepness of the flight path? (Number and/or visual model)

Flight Test

Directions:

1. Toss your airplane 3 to 5 times from some given altitude (or height). Write it down.

2. Measure the horizontal or "ground" distance your airplane flew, using a nonstandard unit.

3. Use your best flight and determine your glide ratio (height:distance)

Glide Ratio (h:d)	Picture of flight path (triangle)	Angle (guess)	Angle (measure)

Hang Glider

Pretend your airplane is a hang glider that has the same glide ratio as your airplane.

1. How much ground distance does the glider cover from jumping off a cliff of . . . 12 meters, 25 meters, 53 meters, 187 meters

Use the ratio table to organize your data.

Ratio Table						
Height	12					
Ground Distance						

Visual Flight Pattern

One hang glider we measured could cover 96 meters on the ground when jumping off a 12 meter cliff.

Draw a side view of the flight path for this glider.

Complete the following ratio table with the data from the new glider:

Ratio Table						
Height		24	96			
Ground Distance				228	1000	

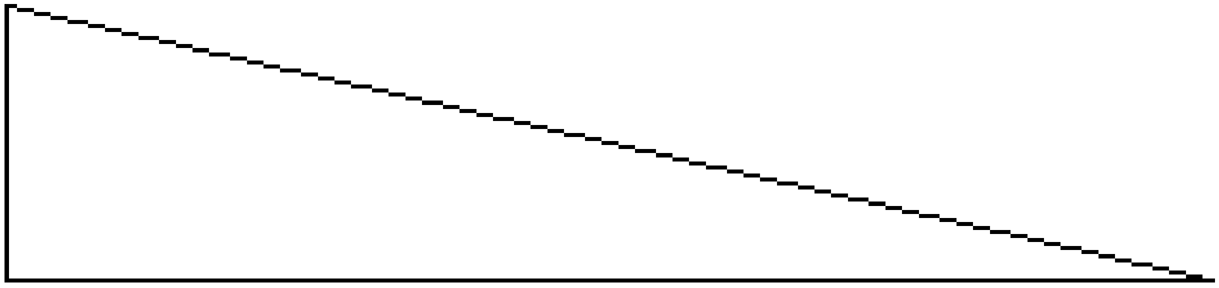
Rachel's Paper Airplane

Here is the flight path of Rachel's airplane.

1. What is its glide ratio?

2. Does it fly farther than yours? Mathematically explain why/why not?

3. How far does Rachel's airplane fly if launched from an altitude of 3 meters?



Glide Ratios

If a glider has a ratio of 1:8 and flew 120 meters on the ground, what altitude did it take off from?

Make a scale drawing of the following glide ratios. Use the circle compasses to determine the angle measure of each.

1:1 1:2 1:4 1:10 1:20

Tangent and Steepness

Steepness in the case of airplanes and gliders is the ratio or comparison of the height and the ground distance. If you divide these two numbers you get the tangent.

Use the tangent table or calculator (to be exact) to convert the ratios on the previous page to an angle measure in degrees.

Angle	Tangent	Angle	Tangent	Angle	Tangent
0	.000	3	.052	6	.105
9	.158	12	.213	15	.268
18	.325	21	.384	24	.445
27	.510	30	.577	33	.649
36	.727	39	.810	42	.900
45	1	48	1.111	51	1.235
54	1.376	57	1.540	60	1.732
63	1.963	66	2.246	69	2.605
72	3.078	75	3.732	78	4.705
81	6.314	84	9.154	87	19.081

Tangent

Find the glide angles for each of the glide ratios below:

Ratio Table						
Glide Angle (alpha)						
Glide Ratio (h:d)		1:1	1:2	1:4	1:10	1:20

Glide ratios can also be expressed as percents. Add the percentages to the next row of the ratio table.

In road construction this is called the road's **"grade"**.

Tangent and Steepness

If your airplane is thrown from an altitude of 15 feet and has a glide ratio of 1:22, then . . .

1. What is the steepness of your airplane's glide path? (Use a circle compass and the ratio and tangent table)

2. What is the grade of your flight path?

3. How far does your airplane fly on the ground?

4. How far does your airplane actually fly?

The Safest Glider

Which of the following gliders is the safest or can fly the farthest?

Glider 1	1:27
Glider 2	0.05
Glider 3	3/78
Glider 4	5 degrees
Glider 5	4% Grade

Write a conjecture regarding the glide angle and the safety or quality of the glider?

Extension Tasks

1. If the angle is 35 degrees, how much ground distance does a glider cover from a height of 100 meters?

2. Why does tan 45 degrees = 1?

3. What angle has a tangent of 2? 3? 4?

4. How much does the measure of the angle change when the tangent value changes from:

0 to 1	1 to 2	2 to 3	3 to 4	4 to 5

Tangent Task

A glider with a glide ratio of 1:28 is launched after being pulled by an airplane to 1,200 meters above Lake Havasu City in Arizona.

1. Indicate on the map how far the glider can fly if there is no wind.

2. How far did the glider fly in the air?

Pyramids and Cube

Put 6 pyramids together to form a cube.

(Use template 7 to build the pyramids. If in groups, combine 6 of them.)

1. What is the formula for volume of a cube?

2. What is the formula for a square pyramid?
    a. How does the formula match the physical model?

Spider Task

What is the shortest path for a spider to crawl from the top corner of the room to the opposite (diagonal) corner of the room?

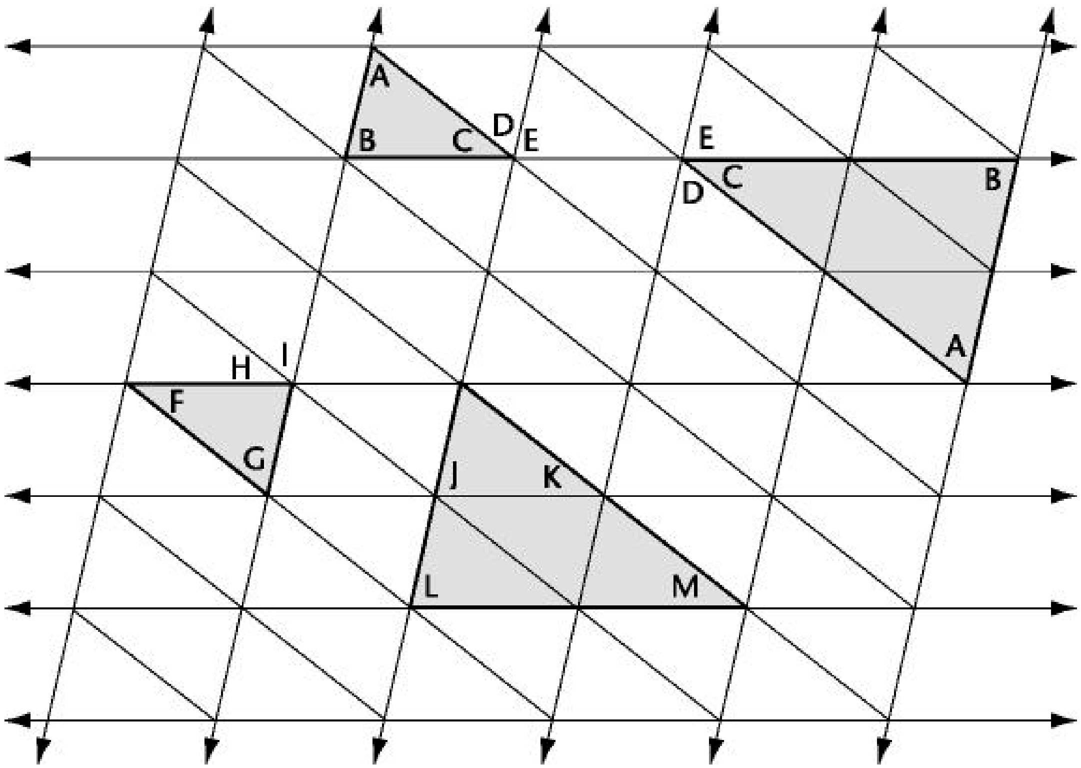
Show your work below.

Transformation Geometry

How do describe the different shapes in the figure below?

Which figures are congruent? Which are similar?

How would describe a translation (slide), reflection (flip), rotation (turn), and a dilation (stretch/shrink)?



MATERIALS

For instructor

- Post it chart paper
- Markers
- Projector (for ppt)

For teachers

- Workbook
- Paper
    - Patty paper
    - Index cards
    - Graph paper
- Tools
    - Scissors
    - Circle protractors
    - Straight edge
    - Compasses
- Other
    - String

PROFESSIONAL DEVELOPMENT MODEL: TRANSFORMING KNOWLEDGE & PRACTICE

In our consideration of the design of the this class, we attempt to "weave together" current ideas about teacher knowledge (subject matter, student thinking, and instruction) with ideas about teacher learning and effective professional development. We describe these ideas in the following paragraphs.

Transforming Knowledge

To accomplish our initial goals, teachers need to possess knowledge of mathematics knowledge about how students learn mathematics, as well as instructional practices that support quality mathematics teaching (Ball & Cohen, 1999; Borko, 2004; Wilson & Berne, 1999). Furthermore, according to recent developments in teacher learning theory, mathematics content and pedagogical content knowledge are playing a greater role in professional development (Loucks-Horsley, et al, 2003).

While knowledge of mathematics content may be at the root of some of the challenges elementary school teachers face, secondary and middle school teachers are more likely to face a different, but related, kind of deficit. Secondary teachers may be competent with common mathematical knowledge, however, reformed teaching requires a special type of knowledge that is grounded in the specific acts of teaching (Ball, Hill, & Bass, 2005 – presentation). Common content knowledge is defined as the procedural and conceptual understandings of mathematics we use to solve mathematical problems and recognize incorrect answers and definitions. Specialized knowledge extends beyond common content knowledge and is the specialized knowledge of mathematics that is particular to mathematics teachers (Ball, Hill, & Bass, 2005). This type of knowledge allows mathematics teachers to analyze and use multiple solution strategies and representations, provide mathematical explanations, and identify misconceptions.

Knowledge of student thinking relates to the knowledge that allows teachers to predict possible student solution strategies and misconceptions and interpret students' ideas. Knowledge of instructional practices is the knowledge teachers use when creating instructional sequences to facilitate student learning such as choosing curricular materials, assessing students, asking questions and reflecting on how to improve their practice (Ball, Thames & Phelps, 2005; Hill & Ball, 2004; Hillet al., 2004).

Ball and colleagues' expansions of the types of teacher knowledge provide a useful framework for teacher professional development and research and are aligned with the principles and major features of teaching for understanding described above. In designing activities to be used in all three of the core classes, we have carefully considered teachers' existing knowledge as well as the knowledge required for teaching, as described above, for early elementary teachers, upper elementary and middle school teachers, and high school teachers.

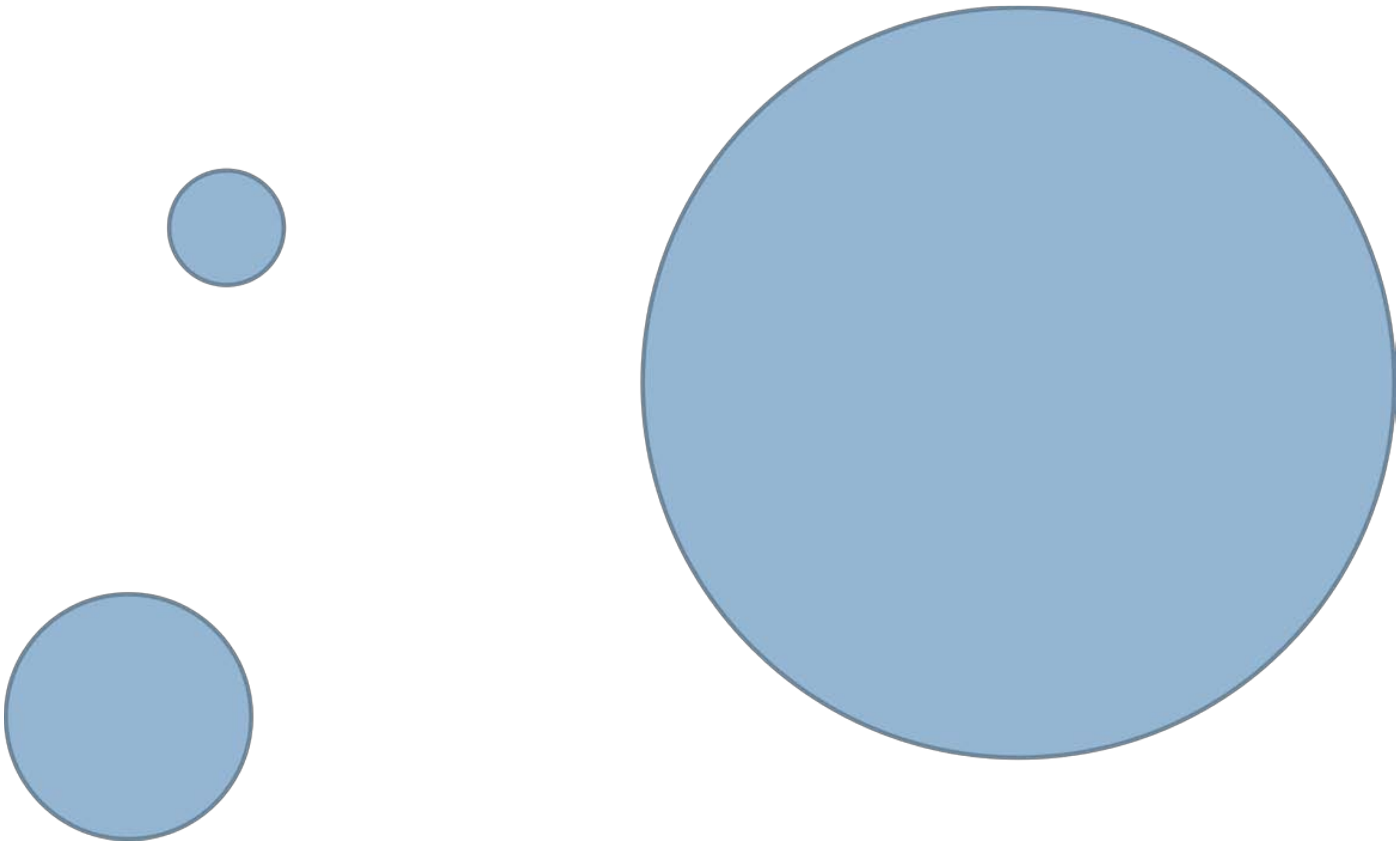
Transforming Practice

While developing knowledge about instruction is crucial, we also consider the need for teachers to engage in the professional activities of teaching, or what Loucks-Horsley et al. (2003) refer to as "building a professional culture." In fact, recent developments in the field of professional development design suggest that collaboration, community building, and participation in communities of learners are key elements in sustaining the impact of high quality professional development (Ball & Cohen, 1999; Borko, 2004; Loucks-Horsley, et al, 2003; Wilson & Berne, 1999).

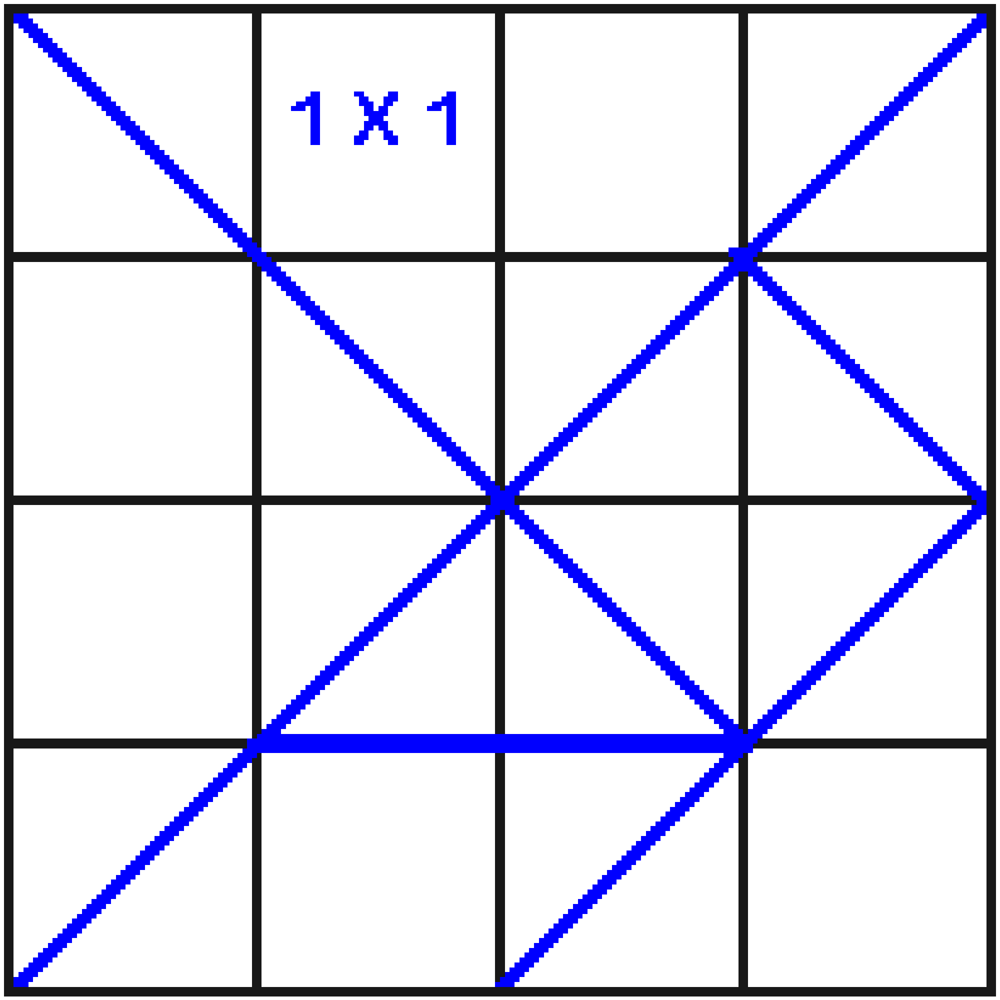
We believe that one key element in learning to participate in a professional community is developing what Ball & Cohen (1999) refer to as a "stance of inquiry." According to Ball & Cohen (1999), effective standards-based instruction is grounded in teachers' ongoing wondering and inquiries about mathematics, student learning and instructional practices. "One way to put the aim here is to help teachers learn the intellectual and professional stance of inquiry – … that would support their generation of multiple conjectures about an issue in practice, their production of alternative explanations, and their efforts to weigh them reactionally" (Ball & Cohen, 1999, p. 27). In other words, teachers should learn to engage in productive discussions and collaborations about their everyday work in classrooms. This type of work should include selecting materials and designing lessons, analyzing student work, wondering about student thinking, and thinking critically about what type of tasks to pose and questions to ask students next.

The knowledge and dispositions required in order to develop a stance of inquiry and to become a productive member of a professional community will be developed through the work in the core class. However, this work will primarily be accomplished with the support of the regional mathematics specialists and instructional leaders within local districts and schools.

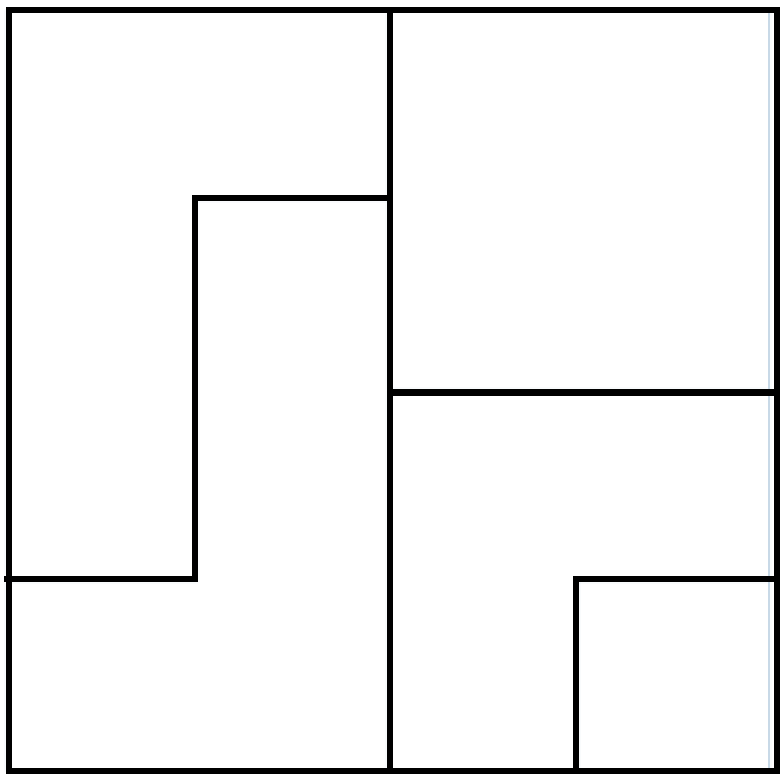
TEMPLATE 1 – CIRCLES

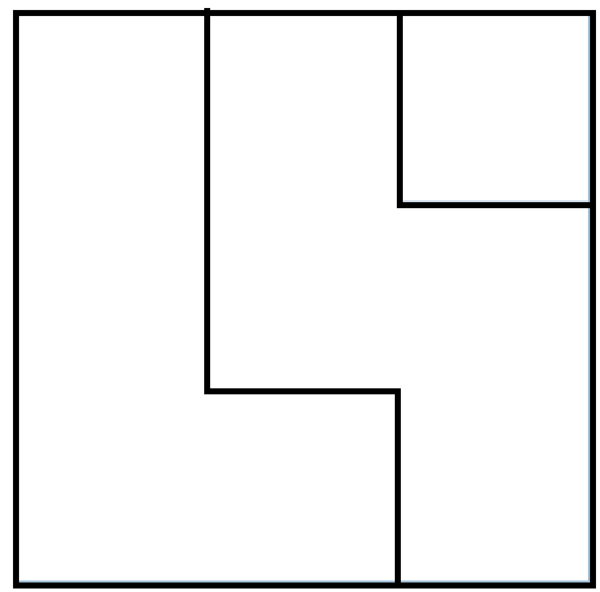


TEMPLATE 2 – TANGRAM

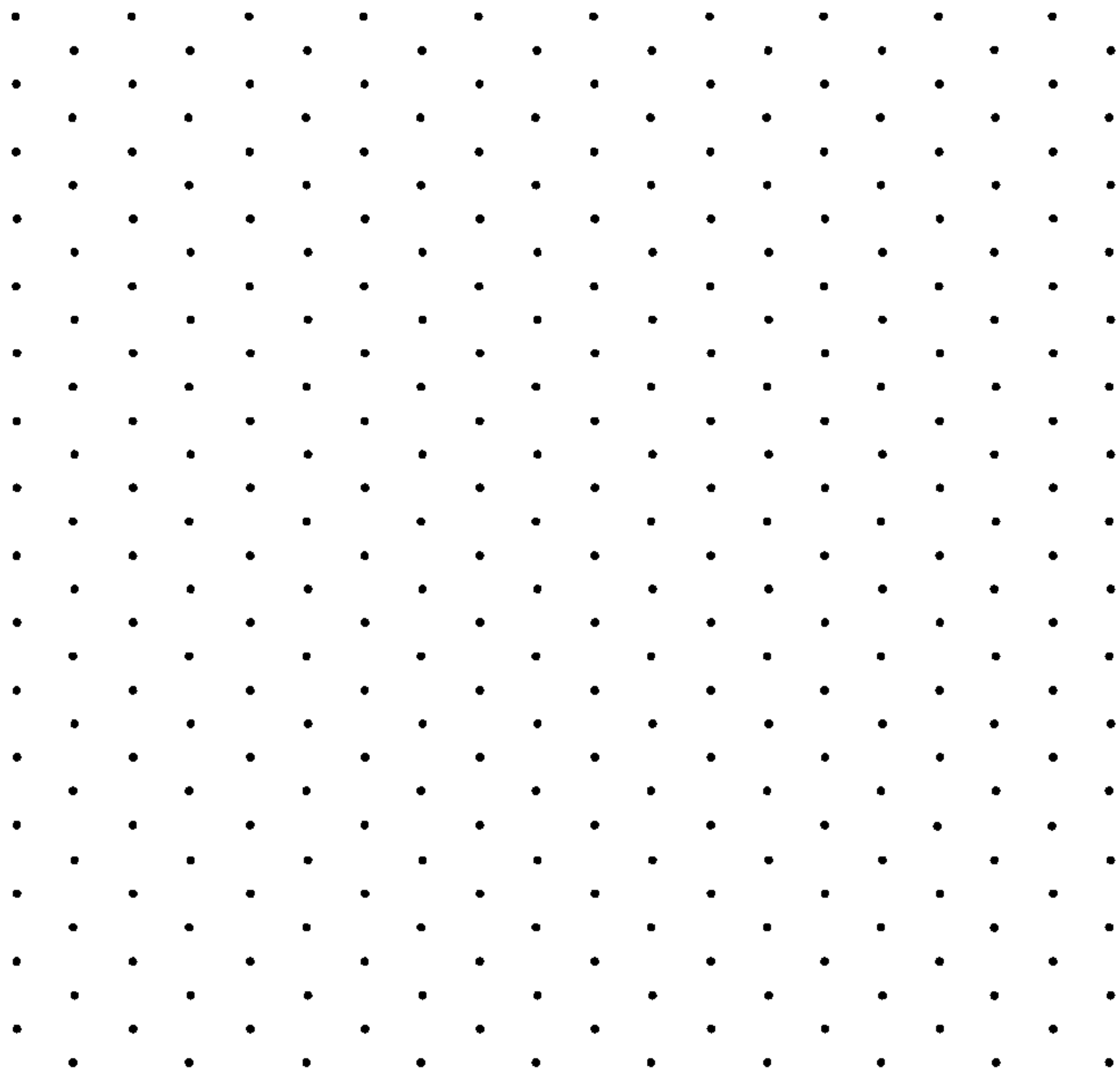


TEMPLATE 3 – COMPOSING SHAPES

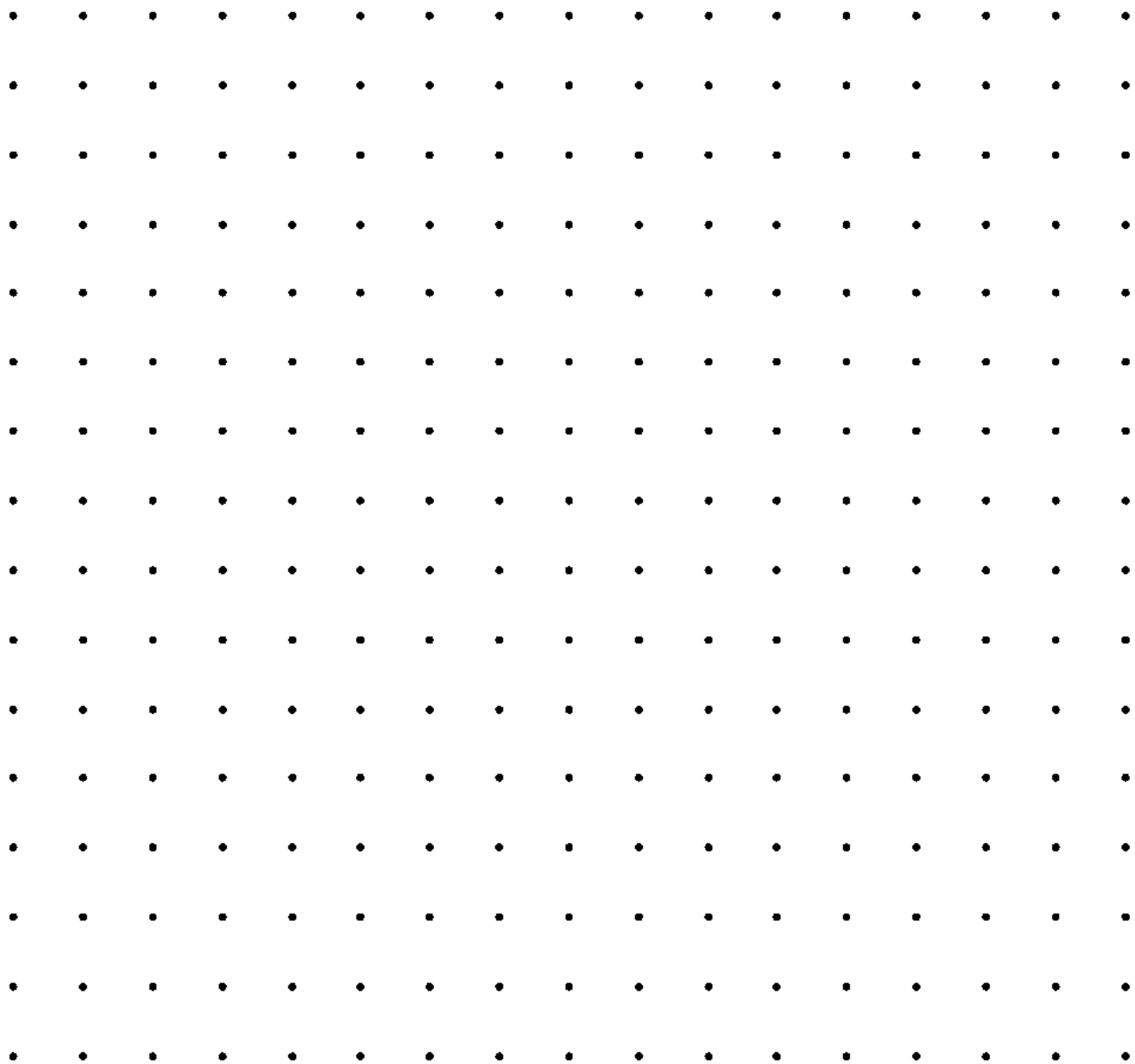




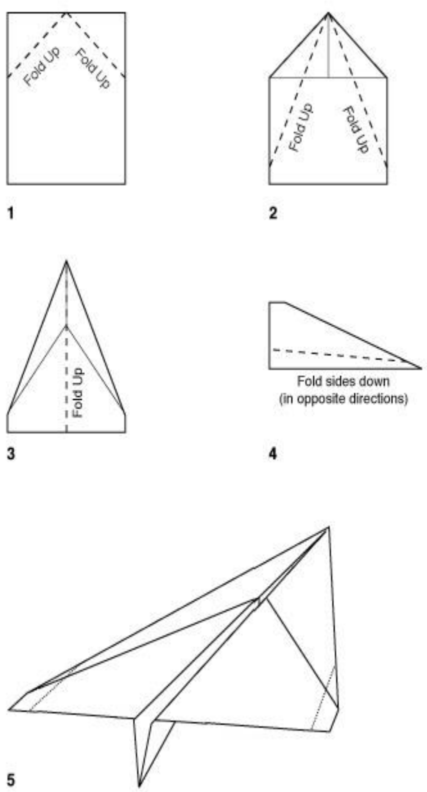
TEMPLATE 4 – ISOMORPHIC DOT PAPER

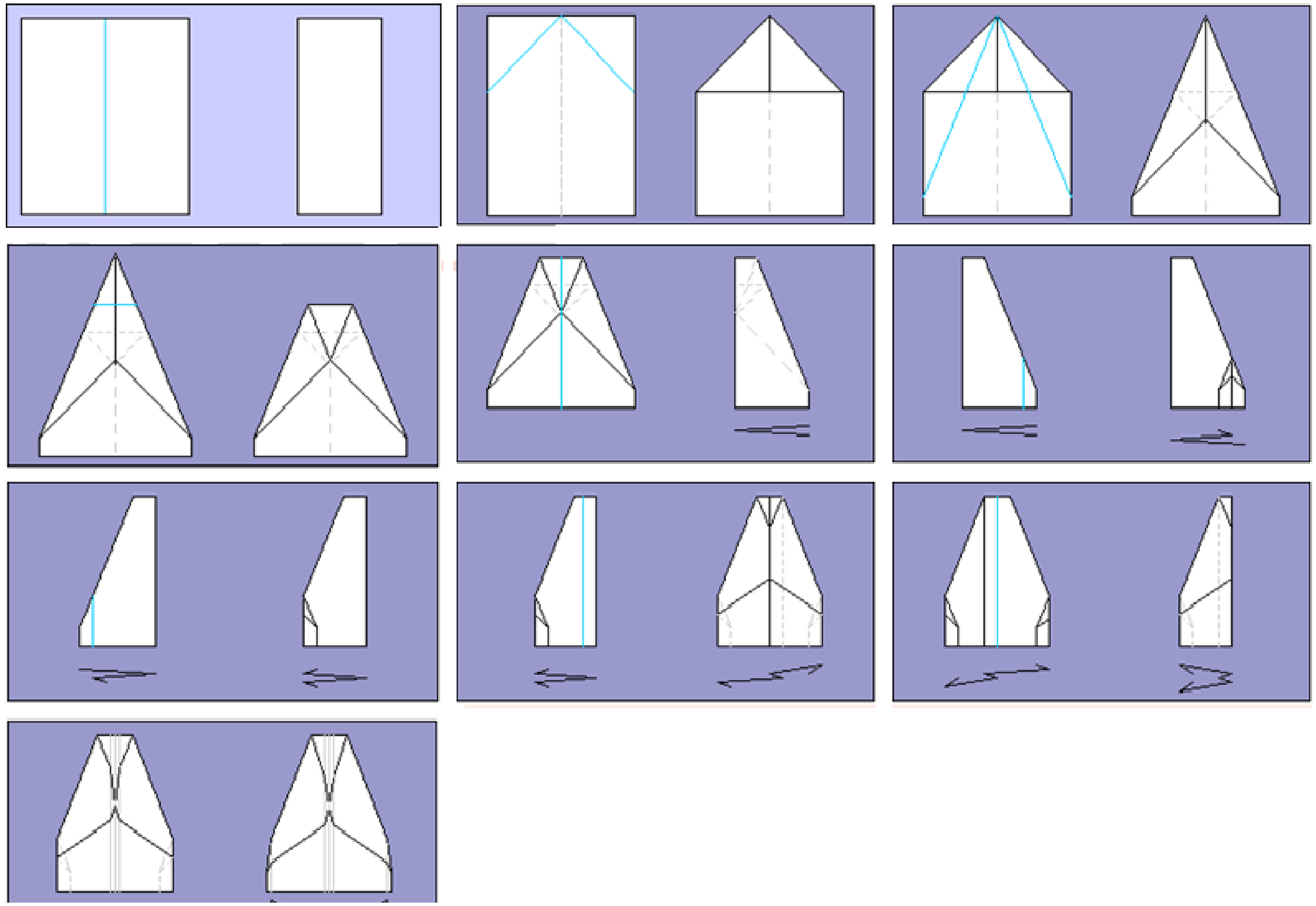


TEMPLATE 5 – DOT PAPER

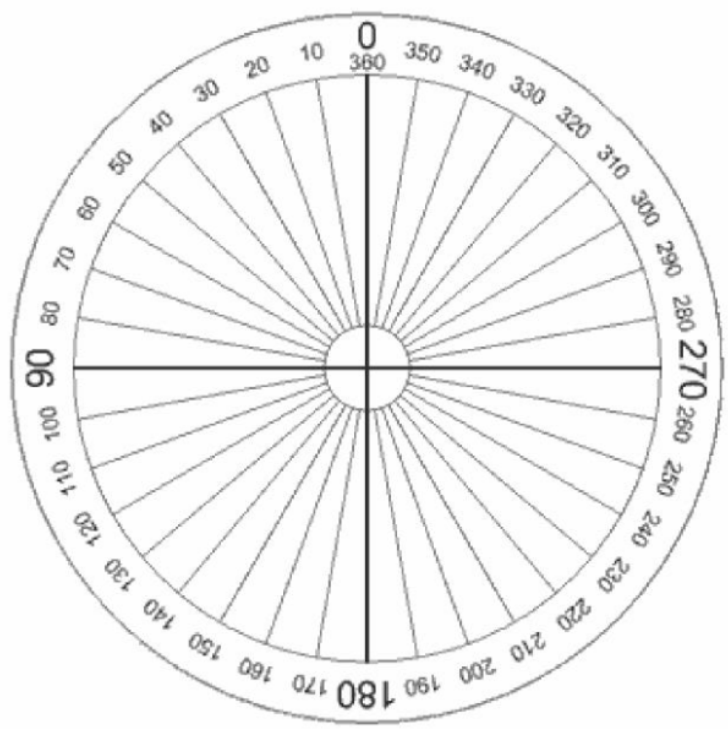


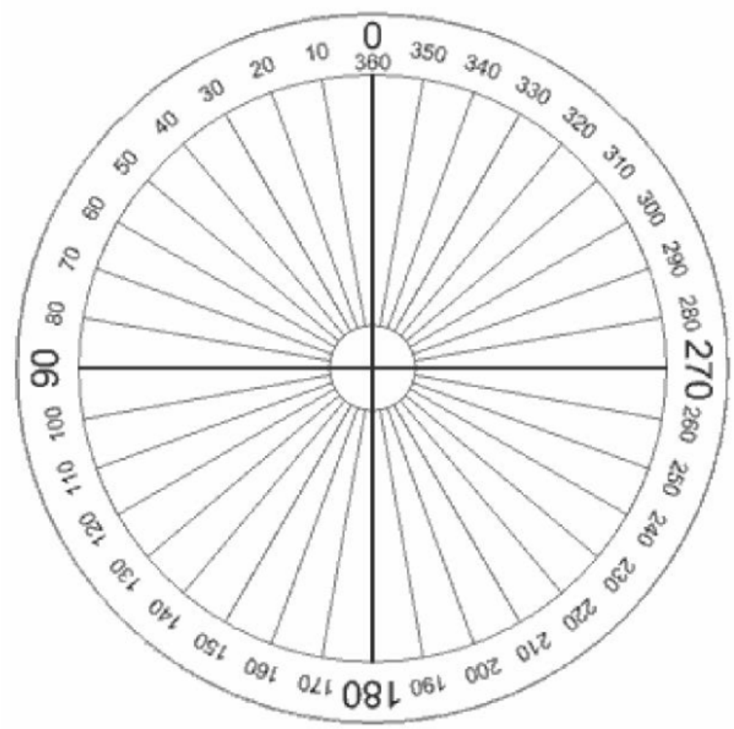
TEMPLATE 6 – AIRPLANES



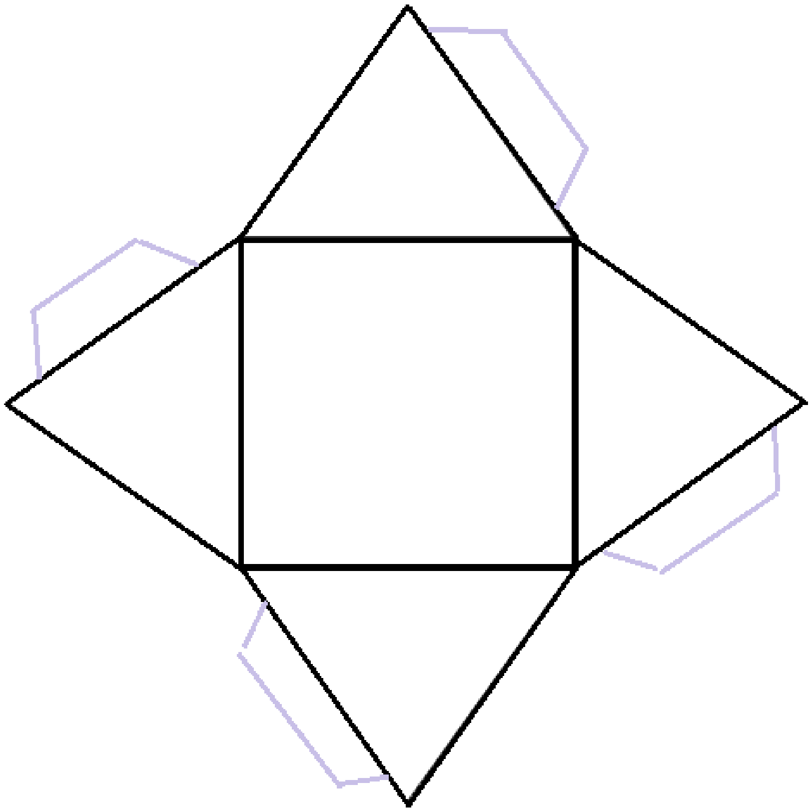


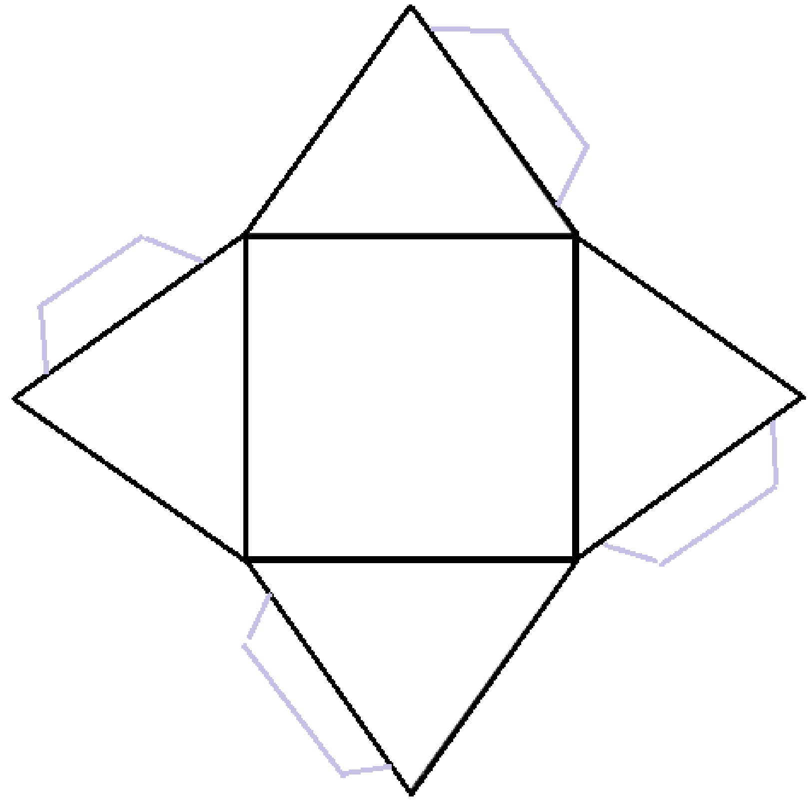
TEMPLATE 7 – CIRCLE COMPASSES





TEMPLATE 8 – SQUARE PYRAMID NET





TEMPLATE 9 – GRAPH PAPER