



DEVELOPING MATHEMATICAL THINKING

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GEOMETRY – AGENDA

Welcome, announcements, housekeeping
Theory of Instruction and PD goals
Chocolate Milk Problem
DMT
Geometry

TEACHING MATHEMATICS

“One important purpose of mathematics education is to prepare students to incorporate mathematical reasoning and communication into their everyday lives. However, conventional pedagogy has often persuaded students to consider school mathematics as a subject divorced from their everyday experiences and from their attempts to make sense of their world.”

Tate, William F. Race, retrenchment, and the reform of school mathematics. Phi Delta Kappan. 1994



*“We construct our **knowledge** of our world from our perceptions and experiences, which are themselves mediated through our **previous knowledge**. Learning is the process by which human beings adapt to their **experiential** world.” [\[1\]](#)*

[\[1\]](#) Simon, Martin. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*.



PROFESSIONAL DEVELOPMENT GOALS

- The professional development goals are to encourage you to provide instruction that:
 - Uses **reasoning** to make sense of problems;
 - Moves from **informal** ideas to more **formal** and abstract ones;
 - Uses **articulation** of one's ideas and conjectures as a focal point to improved understanding.
 - Understands the **generative** process of students' mathematical ideas and how to encourage fluid growth of these ideas.



WHY A MATH INITIATIVE?

- In an international assessment of mathematics performance conducted in 40 countries across the world, the US ranked 28 in 2003 (PISA)
- In 2009, 65 countries participated across the world, and the US ranked 31, which was below the OECD average (PISA)
- Interest in math: math majors at four-year colleges has declined nearly 20% over the past eight years (Boaler, 2008)

WHY PROFESSIONAL DEVELOPMENT?

“A teacher of mathematics has a great opportunity. If he fills allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of the students by setting them to solve problems with *stimulating questions*, he may give them a taste for, and some means of, independent thinking” (Polya)

MENTAL MATH ACTIVITY

Solve this problem mentally, any way you like.
Please do not use paper & pencil.

$$73 + 48$$

After you've solved them, record your strategy using paper & pencil. Try and avoid using any traditional procedures (“algorithm”) to check your answer.

MENTAL MATH ACTIVITY

Solve this problem mentally, any way you like.
Please do not use paper & pencil.

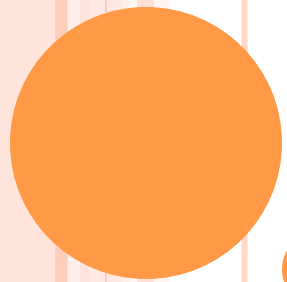
$$57 - 18$$

After you've solved them, record your strategy using paper & pencil. Try and avoid using any traditional procedures (“algorithm”) to check your answer.



CHOCOLATE MILK

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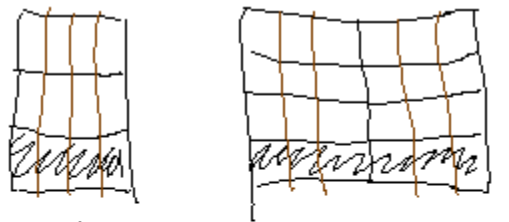


CHOCOLATE MILK PROBLEM

- One glass of milk has $\frac{1}{3}^{\text{rd}}$ chocolate syrup. Another glass that is twice as large as the first glass contains $\frac{1}{4}^{\text{th}}$ chocolate syrup. If you put them together, how much syrup is in the new mixture?



STRATEGY 1



all together $\frac{4+3+3}{36} =$

$$\frac{10}{36} = \frac{5}{18}$$

1



STRATEGY 2

Percentage problem – the entire quantity is 100% or 1.

So, the first glass has $33\frac{1}{3}\%$ syrup and the second glass has $66\frac{2}{3}\%$ syrup (with the remaining % milk).

$$33\frac{1}{3}\%(1/3 \text{ syrup}) + 66\frac{2}{3}\%(1/4 \text{ syrup}) = 100\% \text{ syrup}$$

$$(1/3 * 1/3) + (2/3 * 1/4) = 1/9 + 2/12$$

$$1/9 + 1/6 = 2/18 + 3/18 = 5/18$$



STRATEGY 3

Average

$$33 \frac{1}{3}\% + 25\% + 25\%$$

$$8 \frac{1}{3}\% + (25\% + 25\% + 25\%)$$

$$2 \frac{1}{3}\% + (27\% + 27\% + 27\%)$$

$$7\frac{1}{3}\% \text{ or } 21\frac{1}{9}\%$$

$$27 \frac{7}{9}\%$$

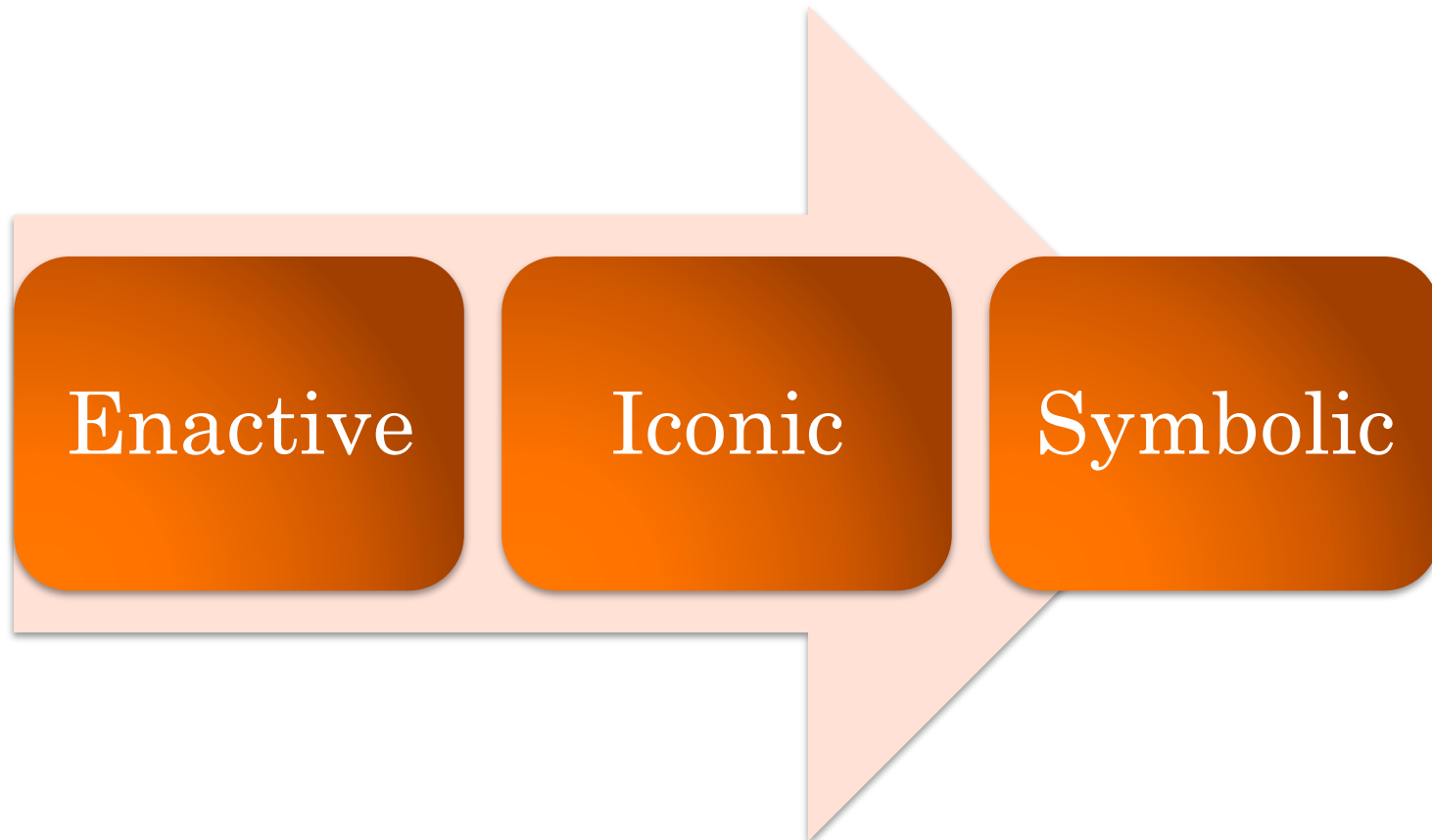
$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} = 3 \text{ cups}$$

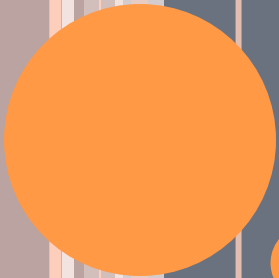
$$\frac{5}{6} = 3x$$

$$X = \frac{5}{18}$$



MODES OF REPRESENTATION



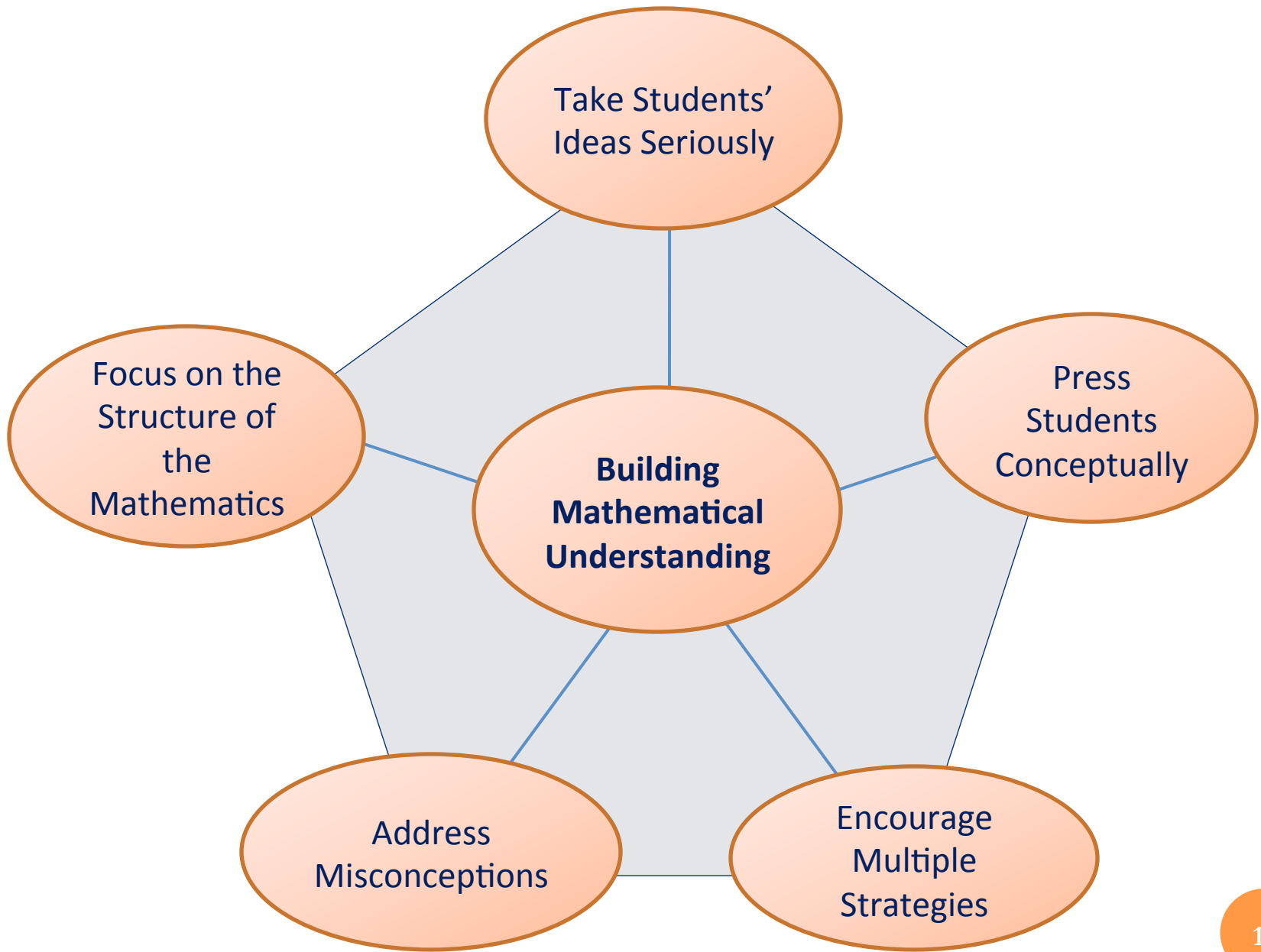


TFU



Teaching For Understanding





TAKING STUDENTS' IDEAS SERIOUSLY: CENTRALITY

- It is the idea that mathematics pedagogy should be built on the experiences of the student;
- Takes the perspective of positioning students within the context of their own lives, experiences, and cultures;
- Centers students in the process of knowledge acquisition;
- Provides students with the opportunities to solve problems using their experiences (from both in & out of school)

PRESS STUDENTS CONCEPTUALLY

- Students must think of how the mathematics work.
- What are the boundaries or limits of the procedures one uses?

ENCOURAGE MULTIPLE STRATEGIES

- Knowing multiple approaches to solving a problem allows one to solve more problems and transfer this knowledge to new situations.

ADDRESS MISCONCEPTIONS

- When one idea is misapplied in another area a student begins to build a misconception.
- A student has a weak understanding of the mathematics when one has a misconception and the best way to get rid of it, is to address when, where, and why it works for one situation and not another.

FOCUS ON THE STRUCTURE OF MATHEMATICS

- What ideas in mathematics cross multiple areas?
- How are ideas connected to other ideas?

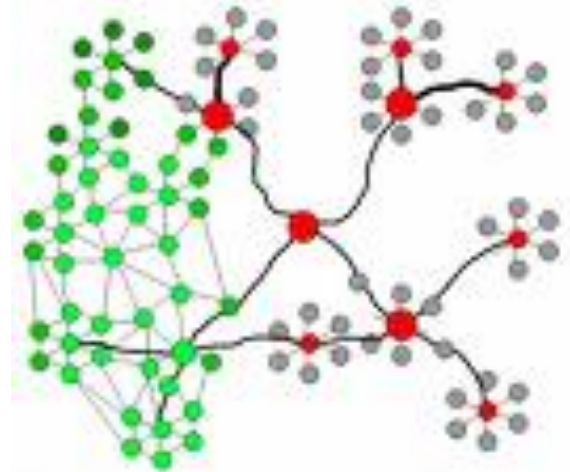
INSTRUCTIONAL MODEL: TFU

- Teaching For Understanding
- Connecting related pieces of knowledge together
- Knowing *how* to do something and *why*
- Knowing how . . .
 - Enables us to flexibly use procedures and understand the relationship of these procedures within the structure of mathematics
- Knowing why . . .
 - Enables us to use concepts flexibly, extend our knowledge to new situations, and connect it to the world outside of school

TEACHING FOR UNDERSTANDING

- Structural Perspective

- Knowledge is structured through web-like/ hierarchal connections.
- Mental representation as part of a network of representations.
- **Claim:** The stronger and the greater number of connections there are in this complex structure, the higher degree of understanding.



TEACHING FOR UNDERSTANDING

- Functional Perspective

- Maintains that students need to actively integrate incoming information with existing knowledge through social interactions.
- So, students must interact and share knowledge with others.
- **Claim:** By being in situations in which students are communicating with others, they build lasting and coherent concepts and skills, which leads to an increased degree understanding.

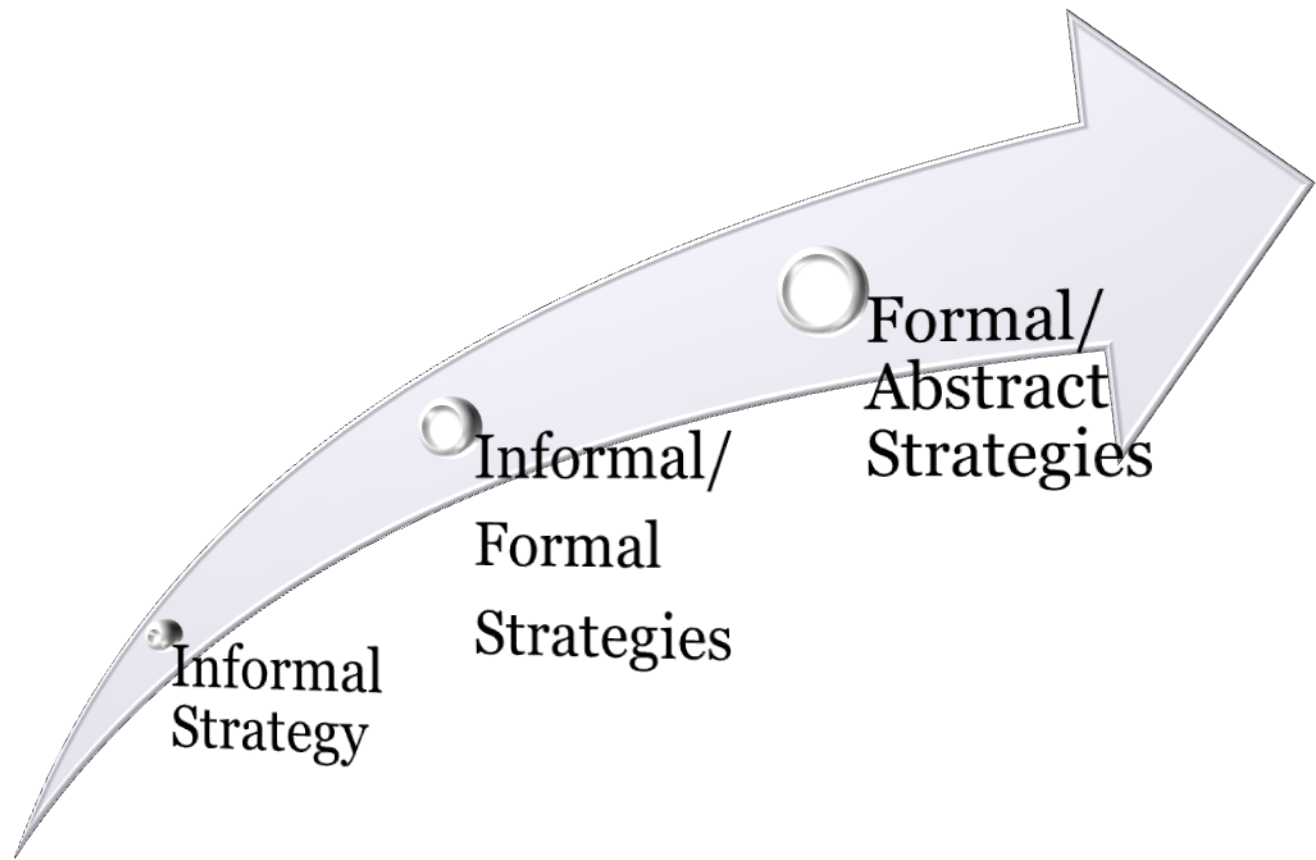
BUT *HOW* DOES ONE DO THIS?

- Realistic Mathematics Education
(Gravemeijer & van Galen, 2003;
Freudenthal, 1973, 1991; Treffers, 1987)
 - Guided Reinvention
 - Mathematizing

INSTRUCTIONAL PRACTICE: GUIDED REINVENTION

- Is the process of first allowing students to **develop informal strategies** for solving problems, and then, by critically examining those strategies, encouraging students to **develop more sophisticated**, formal, conventional and abstract strategies and algorithms
- Students are encouraged to make connections between existing knowledge (informal ideas) and new knowledge (more formal mathematical ideas) – structural aspect of understanding.

GUIDED REINVENTION: TRAJECTORY



GUIDED REINVENTION: ARITHMETIC

“By thinking and talking about similarities and differences between arithmetic procedures, students can **construct relationships** between them. ... the instructional goal is not necessarily to inform one procedure by the other but, rather, to **help students build a coherent mental network** in which all pieces are joined to others with multiple links.”

MATHEMATIZING (TREFFERS, 1987)

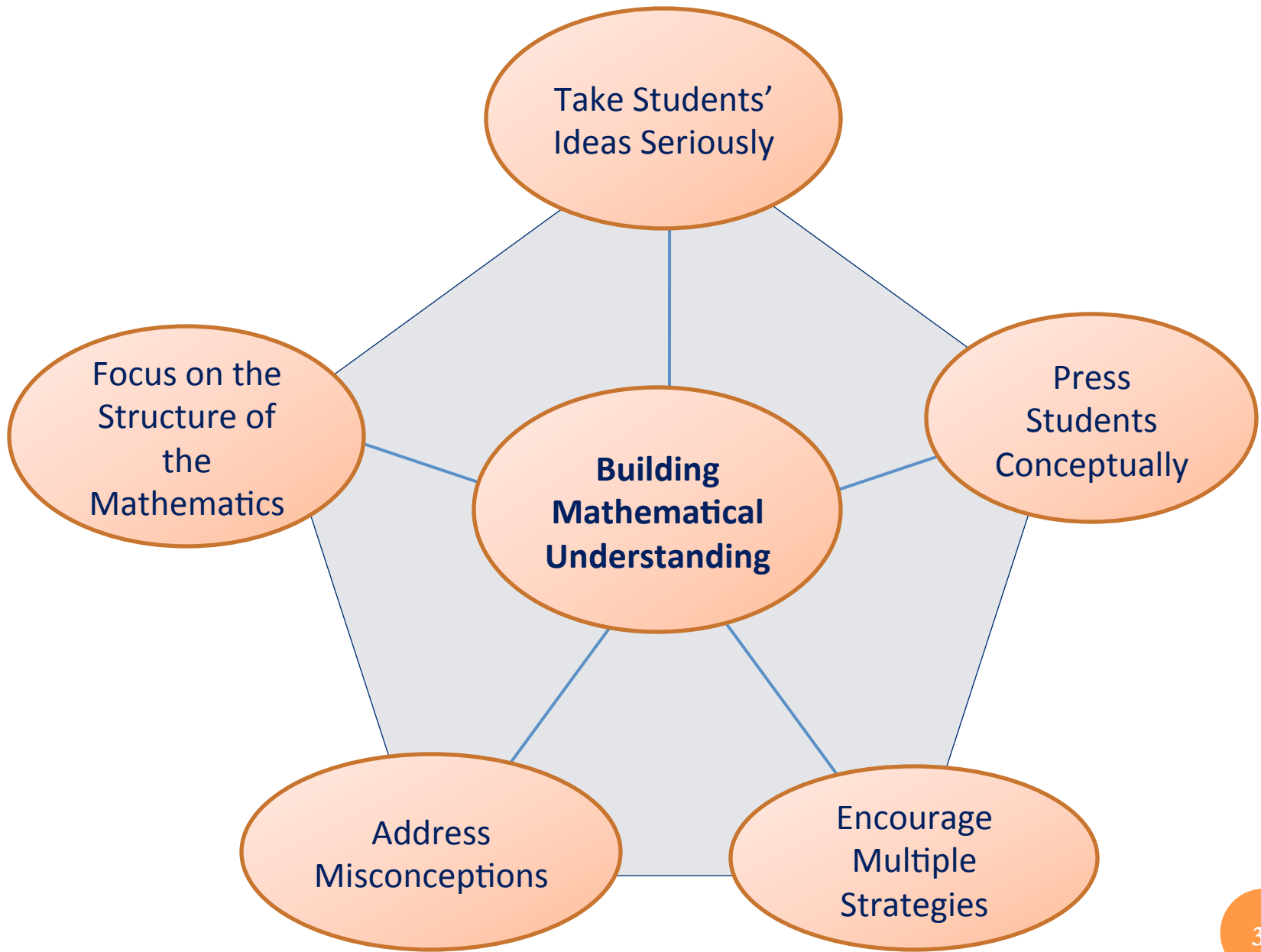
- Horizontal Mathematization
 - Occurs when students represent a contextualized problem mathematically in order to find a solution strategy.
- Vertical Mathematization
 - Involves taking the mathematical matter to a higher level, and occurs when students make their representations and strategies objects of mathematical examination.

MATHEMATIZING & GUIDED REINVENTION

Guided reinvention ‘happens’ when students mathematize.

Students should go through a similar process as mathematicians go through themselves.

Objective: “...for students to experience their mathematical knowledge as the product of their own mathematical activity” (Gravemeijer & van Galen, 2003, p. 117)



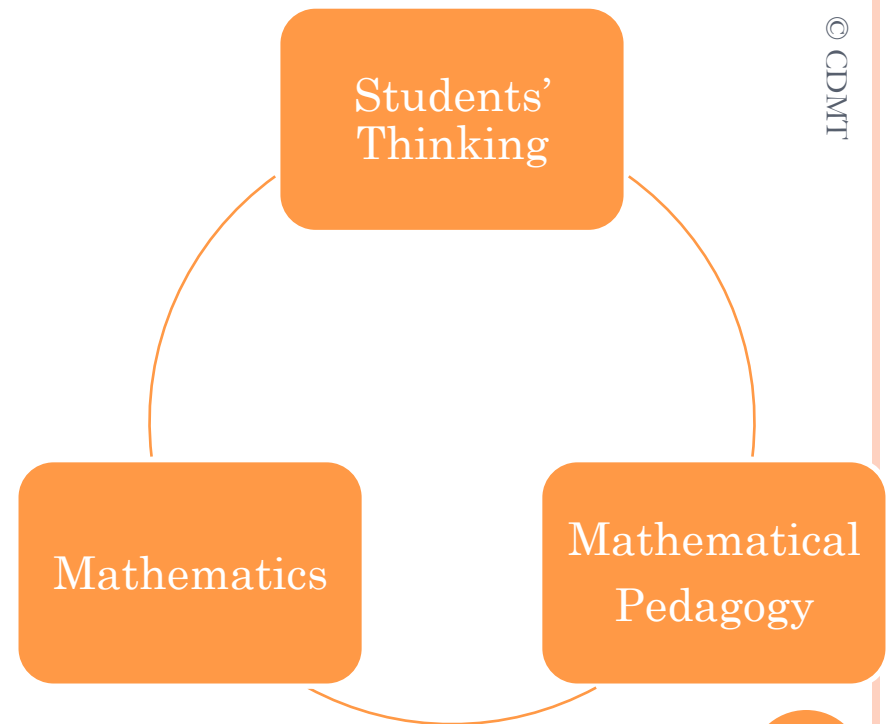


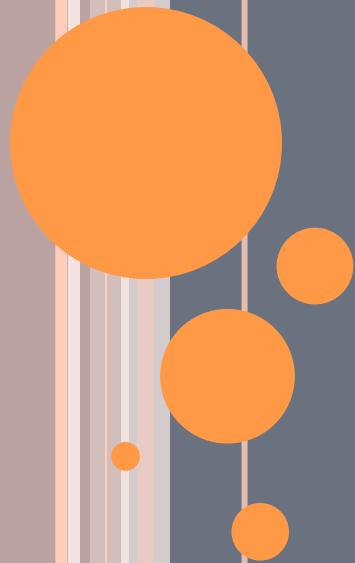
PD GOALS

34

PROFESSIONAL DEVELOPMENT GOALS

- Teachers must know not only the formalism of mathematics but the informal and generative process of students' mathematical ideas and how to encourage fluid growth of these ideas.





26 - 19

CONTEXTUALIZED PROBLEMS

- Alexander walked 26 steps. Maria walked 19 steps. How many more steps did Alexander walk than Maria?



CONTEXTUALIZED PROBLEMS

- Alexander walked 26 steps. Maria walked 19 steps. How many more steps did Alexander walk than Maria?
- Alexander walked 26 steps forward and then took 19 steps backwards. How many steps away from the start is he?



COMPARE DIFFERENCE UNKNOWN

- Alexander walked 26 steps. Maria walked 19 steps. How many more steps did Alexander walk than Maria?
- Pressing students to verbalize and mathematize:
 - How did you solve the problem? (verbalize)
 - **Horizontal mathematizing** – mathematically model the problem
 - How can you express your answer mathematically; how do you notate it?
 - Does your model/notation MATCH . . .
 - how you solved the problem? (strategy)
 - how the problem was delineated in the story format? (context)



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(19) 20, 21, 22, 23, 24, 25, 26
1 2 3 4 5 6 7/11



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$$19 \xrightarrow{+1} 20 \xrightarrow{+6} 26$$

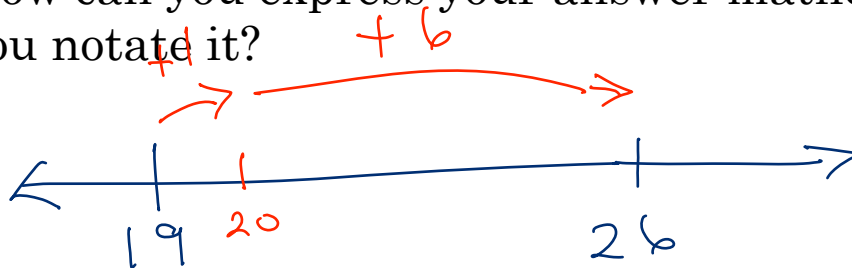
So $1 + 6 = \underline{\underline{7}}$



COMPARE DIFFERENCE UNKNOWN

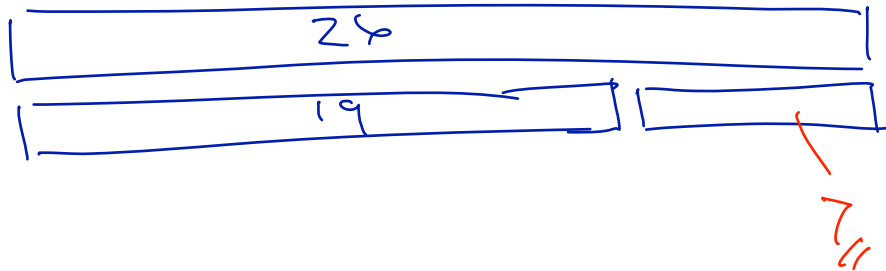
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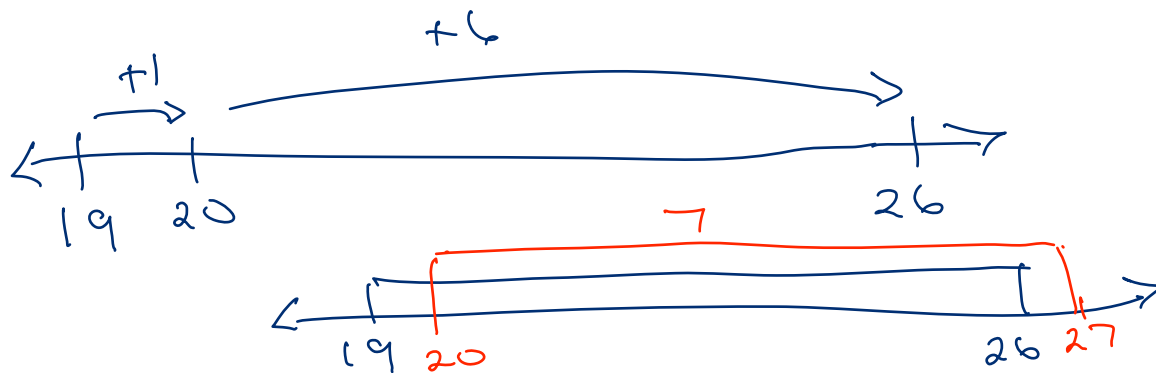


COMPARE DIFFERENCE UNKNOWN

Alexander walked had walked 26 steps. Maria walked 19 steps. How many more steps did Alexander walk than Maria?

- Pressing students to verbalize and mathematize:
 - How are these two models similar; different? (verbalize)
 - **Vertical mathematizing** – reification; using the model as an object of discussion
 - Why does these model work?

19 20, 21, 22, 23, 24, 25, 26

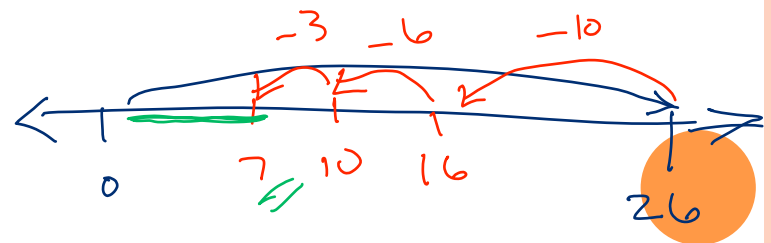
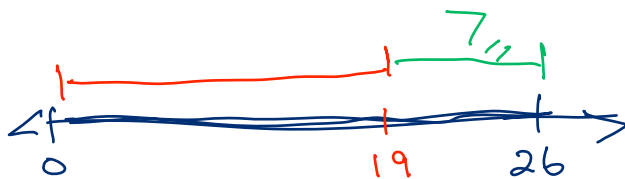


VERTICAL AND HORIZONTAL MATHEMATIZING

Alexander walked 26 steps. Maria walked 19 steps. How many more steps did Alexander walk than Maria? (CDU)

Alexander walked 26 steps forward and then took 19 steps backwards. How many steps away from the start is he? (SRU)

- Does your model/notation MATCH . . .
 - how you solved the problem? (strategy)
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